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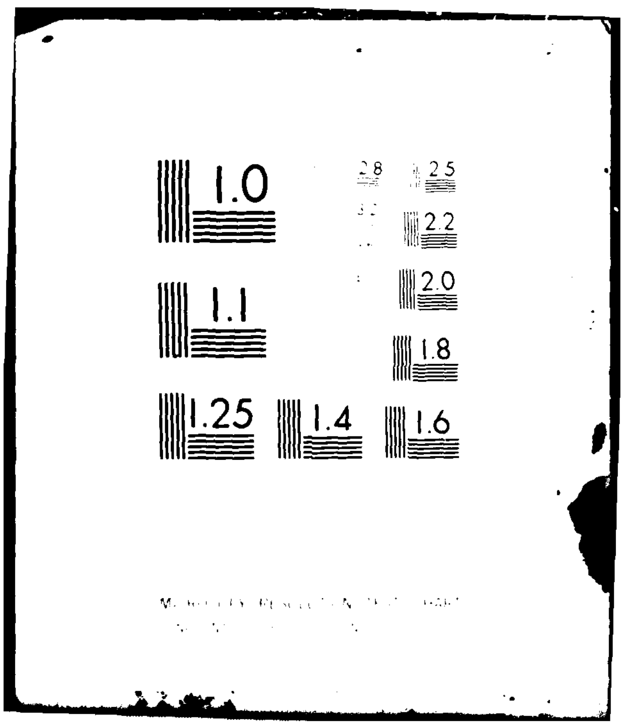
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UNSTEADY TRANSONIC FLOWS IN TWO-DIMENSIONAL CHANNELS WITH OSCILLATING BOUNDARIES

T.C. ADAMSON, JR.
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October 1981

Final Technical Report
Prepared for
Naval Air Systems Command
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Unsteady transonic flow in a two-dimensional duct with a shock wave is considered. The flow entering the duct is uniform and steady; unsteadiness is caused by any mixture of impressed oscillations in wall shape and back pressure. Asymptotic methods are employed to give solutions for flow properties and for position and velocity of the shock wave in several different problems, characterized by the relative orders of the nondimensional

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) amplitude and frequency of the oscillations. Illustrative example flows are considered, with numerical solutions used for complex wall shapes.

UNSTEADY TRANSONIC FLOWS IN TWO-DIMENSIONAL CHANNELS WITH OSCILLATING BOUNDARIES

by

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I. INTRODUCTION

This report is concerned with the analysis of unsteady two-dimensional transonic flow in channels. Although the motivation behind the work was the study of the effects of oscillating walls on the flow, it was found that in the problems studied it was possible to include the effects of oscillations in the back pressure, interpreted as pressure variations impressed upon the flow at some point downstream of the shock wave. Hence either of, or any mixture of, the two means of causing unsteadiness in the flow may be considered; of particular importance is the ability to describe the kinds of wall motion which correspond to a given oscillation in back pressure. Because both modes of introducing unsteadiness are retained, the results have applications to the flow in inlets and to the flow between the compressor blades in turbojet engines and to the diffusers of ramjet engines, as long as the flow is in the transonic regime.

Physically, wall oscillations may be associated with either blade flutter or the displacement of the flow due to a turbulent boundary layer or a separation bubble; the oscillations in back pressure are associated with pressure fluctuations caused by combustion instabilities or interactions with the wakes of blades or flow asymmetries, etc., downstream of the channel in question. When the relative velocity at the tip of a rotor blade exceeds the sonic value, the possibility of supersonic unstalled flutter has been demonstrated [1,2]. In addition, interaction between the core flow and the boundary layer has been observed in experiments involving transonic flows through channels with separation [3,4]; apparently, self-induced oscillations occurred with variations in the position of the shock wave and the location and size of the separation bubble. Although these two phenomena are examples of cases where the effective boundary of the inviscid flow field moves with time, it is clear that changes in back pressure could certainly affect the interactions. Finally, in a ramjet engine (and conceivably in a turbojet engine) combustion instabilities could propagate upstream and thus be the source of the oscillations in pressure [5]. Indeed, in a ramjet engine, it appears possible that self-sustained oscillations could be generated as a result of the interaction between the diffuser and the combustion chamber; when an internal shock wave occurs in the diffuser, such oscillations could lead to this shock being disgorged and thus to diffuser "buzz" or even more severe problems.

The experimental work done at McDonnell Douglas (e.g., Refs. [4] and [5]) has led to analyses of viscous-inviscid interactions in transonic internal flows, both unsteady [6] and steady [7], but with unseparated flow; results have not predicted self-induced oscillations and so the more difficult separated-flow problem is now being considered by Liou [8], among others.

The work described herein is concerned with one aspect of the interaction problems mentioned above; as such, it is closely related to the experimental work done on diffuser flows carried out by Sajben et al [4,5] . Specifically, it has to do with the prediction of the inviscid transonic flow field with a shock wave in a two-dimensional channel, using asymptotic methods of analysis. Previous to this work, in studies of unsteady transonic flow in channels [9, 10, 11, 12] involving asymptotic methods, the unsteadiness was caused by pressure oscillations downstream of the shock wave such that time dependence occurred only in second- and higher-order terms; the flow was steady to first order. The solutions found are valid for inviscid flows in symmetric and asymmetric channels with small- and large-amplitude oscillations in shock-wave position; in all cases, the walls are stationary. It can be shown, also, that a transonic flow between the blades of a two-dimensional cascade is described asymptotically as a channel flow. Because the previous channel-flow solutions were written in terms of arbitrary wall shapes, they may also be applied to a two-dimensional cascade flow with arbitrary airfoil shapes under conditions where the blades are stationary and oscillations in pressure originate downstream of the cascade. In the present work, the same range of parameters is considered as in Refs. [9] to [12], but for the case where the unsteadiness could be caused by oscillations in wall shape or back pressure or by any combination of the two.

The fact that the problems considered all involve inviscid flow fields should not be construed as being restrictive insofar as use of the results are concerned. Because the solutions are written in terms of arbitrary wall shapes, the effects of boundary layers may be accounted for by adding their displacement thickness to the actual wall shape; in the range of parameters considered, the boundary layer is quasi-steady. In practice, the calculation is accomplished by setting up a computational procedure in which the boundary-layer displacement thickness and channel-flow conditions are calculated simultaneously. Such a procedure has been used by Liou [7] who, in calculating the pressure distribution, also accounted for the interaction between the shock wave and a turbulent boundary layer using asymptotic techniques developed by Messiter [13]; comparison between experimentally derived and computed pressure distributions showed excellent agreement.

In the next section, the general problem is formulated and the specific problems are defined in terms of the relative orders of the parameters. The following sections contain the solutions obtained during the one-year period of the contract. In the final section, the results are summarized.

II. GENERAL PROBLEM FORMULATION

The problem under consideration is that of an open duct placed in a uniform transonic flow. Inside the duct the cross-section area varies with distance along the duct. On the outside, the duct walls are, for convenience, assumed parallel to the incoming uniform flow. The lack of any cowl effects on the outer flow is not important to the results obtained; such effects are omitted only to make calculations as simple as possible. A sketch of the duct and the notation employed is shown in Fig. 1. The coordinates x and y are made dimensionless with respect to $\bar{L}$, where the total width of the duct is taken to be $2\bar{L}$. (Overbars denote dimensional quantities.) The corresponding velocity components, u and v, are dimensionless with respect to the critical sound speed in the incoming flow, $\bar{a}^*$. Another characteristic dimension of the channel is its length, denoted by $\bar{c}$. The channel flow can be associated with that between the blades in a two-dimensional cascade with zero stagger angle; the length of the channel is then the chord of the blade in this equivalent cascade flow. For the present calculations, the flow at the exit cross-section might be considered to exhaust into a plenum, or the duct might be regarded as joined smoothly to fixed plane walls extending further downstream.

In general, the nondimensional parameters associated with the flow unsteadiness are the characteristic time (or frequency) and amplitude of the impressed oscillations in pressure or in boundary position, made dimensionless with the fundamental parameters of the basic steady channel flow. Thus, if $\overline{T}_{ch} = 1/f$ is the characteristic time associated with walls or back pressure oscillating at a frequency f, then we define

$$\tau \equiv \frac{\overline{T}_{ch}}{\overline{L}/\overline{a}^*} \tag{1}$$

Here, $\overline{L}/\overline{a}^*$ is a characteristic flow or residence time in the channel, because the reference velocity for this transonic flow is $\overline{a}^*$; however, $\overline{L}/\overline{a}^*$ is also the order of the time taken by an acoustic signal to traverse the channel. The parameter τ may be related to the familiar reduced frequency used in flutter analysis, by noting that if $\overline{s} = 2\overline{L} \equiv \overline{c}s$ is the blade spacing (s is dimensionless with respect to the chord, as shown), then,

$$\tau = \frac{2\pi}{s\,k_r} \tag{2}$$

where

$$k_r = \frac{(2\pi f)\,(\overline{c}/2)}{\overline{a}^*} \tag{3}$$

is the reduced frequency associated with the blade motion, based on the half-chord.

The upper and lower wall shapes are written, respectively, as

$$(y_w)_u = 1 - \epsilon^2 f_u(x) + \alpha\, G_u(x,t) \tag{4a}$$

$$(y_w)_\ell = -1 + \epsilon^2 f_\ell(x) - \alpha\, G_\ell(x,t) \tag{4b}$$

$$G_{u,\ell} = \{(x - x_c)\,\beta(t) + G_p(t)\}_{u,\ell} \tag{4c}$$

where $f_u(x)$ ɔd $f_\ell(x)$ ve the steady-state wall shape, with $f_u = f_\ell = 0$

at the leading edges, and t is the time made nondimensional with $\overline{T}_{ch}$. The motions of the upper and lower walls, which may or may not be in phase with each other, are denoted by G_u and G_ℓ, respectively; x_c is the center of rotation, with $\alpha\,\beta(t)$ being the instantaneous angle made by the outer duct wall measured relative to the x axis, and $G_p(t)$ represents a plunging motion independent of x. The thickness of the "blades" or duct walls is then $O(\epsilon^2)$ where $\epsilon^2 << 1$; another way of interpreting ϵ is to note that the nondimensional radius of curvature of the nozzle formed by the duct inner wall at the throat is $O(\epsilon^{-2})$. The parameter α, with $\alpha << 1$, orders the amplitude of the motion of the walls, this depending upon the problem considered. As will be seen, it is the relative orders of α, τ, and ϵ which determine the order of the amplitude of the shock-wave motion and the orders of the velocity and pressure perturbations due to the unsteadiness in the flow. Finally, it may be noted that for a symmetric channel with symmetric wall motions, $f_\ell(x) = f_u(x)$ and $G_\ell(x,t) = G_u(x,t)$.

Since the flow is transonic, the undisturbed-flow Mach number M_∞ differs from one by a small amount. The difference $M_\infty - 1$ is taken here to be $O(\epsilon)$, which is the case of greatest interest because the steady-state variation in channel width is $O(\epsilon^2)$ and so the changes in Mach number within the channel are of the same order as $M_\infty - 1$. Since α is taken to be an integral power of ϵ, the expansions for the velocity components have the form

$$u = 1 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots \tag{5a}$$

$$v = \epsilon^2 v_2 + \epsilon^3 v_3 + \cdots \tag{5b}$$

where u_1, u_2, $\cdots$ and v_2, $\cdots$ are functions of x, y, and t. These representations are correct throughout most of the channel, except perhaps for thin regions close to the leading and trailing edges, the throat, and the shock wave. Corrections for such regions are introduced later when needed.

If the governing equations are written in dimensionless form, one might choose to make the time dimensionless with respect to the characteristic residence time; i.e., $T = \bar{T}/(\bar{L}/\bar{a}^*)$. However, for impressed oscillations on the flow $\bar{T} = O(\bar{T}_{ch})$ and so, as noted earlier, we introduce a time coordinate t made nondimensional with $\bar{T}_{ch}$; then T is related to t by a coordinate stretching as follows:

$$\frac{\bar{T}}{(\bar{L}/\bar{a}^*)} = T = \tau t \qquad t = \frac{\bar{T}}{\bar{T}_{ch}} \tag{6a,b}$$

where τ is defined in Eq. (1).

The relations for the shape and velocity of the shock wave may be written as

$$x_s(T,y) = x_{s0}(t) + \epsilon x_1(t) + \cdots \tag{7a}$$

$$u_s = \frac{dx_s}{dT} = \frac{1}{\tau}\left[\frac{dx_{s0}}{dt} + \epsilon\frac{dx_{s1}}{dt} + \cdots\right] \tag{7b}$$

That is, it can be shown [9] that $\partial x_s/\partial y = O(\epsilon^{3/2})$ and that v_s, the component of the shock wave velocity in the y direction, is of high enough order that it may be neglected. The velocity of the shock wave u_s is positive in the positive x direction. Because the fluid velocity is near the speed of sound and the velocity relative to the shock wave must be supersonic, the velocity of the oscillating shock wave must be small compared to the sonic velocity; i.e., $|u_s| << 1$. Thus, if $\tau = O(1)$, for example, $dx_{s0}/dt = 0$ so that the amplitude of the shock motion cannot be $O(1)$ but is at most $O(\epsilon)$, from Eqs. (7).

It is desired to find solutions for the velocity components valid to $O(\epsilon^2)$. This allows calculation of the thermodynamic variables to order ϵ^2 also. Now, as will be seen, it is necessary to calculate the term of order ϵ^3 in v and to apply the boundary conditions at the wall in order to derive an equation for an unknown term in u_2. Hence, governing equations for third-order terms must be derived. Because the flow is inviscid and starts at uniform conditions, it is isentropic up to the shock wave. Because the flow is transonic, shock waves are weak, and the jump in entropy across the wave is $O(\epsilon^3)$; however, the gradient in entropy, transverse to the flow, is the important term in ordering the vorticity, and this gradient in the y direction is at least one order higher, i.e., $O(\epsilon^4)$, downstream of the shock wave. Therefore, up to and including terms $O(\epsilon^3)$, the flow is irrotational upstream and downstream of the shock wave, so that a velocity potential exists. Thus, one can write a perturbation potential in the form

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots \tag{8}$$

with $u = 1 + \phi_x$, $v = \phi_y$, $u_1 = \phi_{1x}$, $v_1 = \phi_{1y}$, etc. The governing equation for ϕ is, then, the full potential equation [14],

$$(a^2 - u^2)\,\phi_{xx} + (a^2 - v^2)\,\phi_{yy} - 2uv\phi_{xy} - \phi_{TT}$$

$$- 2u\,\phi_{xT} - 2v\,\phi_{yT} = 0 \qquad (9)$$

where a is the speed of sound made dimensionless with respect to $\bar{a}^*$; a is given by the Bernoulli equation for unsteady flow [14] :

$$\frac{a^2}{\gamma - 1} + \frac{q^2}{2} + \phi_T = F(T) \qquad (10)$$

where $q^2 = u^2 + v^2$, and $a^2/(\gamma - 1) + q^2/2 = H$ is the stagnation enthalpy, dimensionless with respect to $\bar{a}^{*2}$; F(T) is an integration function to be determined.

Expansions for the density ρ, pressure P, and temperature $\hat{T}$, each made dimensionless with respect to its critical value (e.g., $\rho = \bar{\rho}/\bar{\rho}^*$) can be written in the same form as that for u, Eq. (5a); thus, for example,

$$\rho = 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots \qquad (11)$$

with similar expansions for P and $\hat{T}$. Also, it is easily shown that $a^2 = \hat{T}$. The property relationship, in dimensionless form is

$$P = \rho^{\gamma}\, e^{\left(\frac{\gamma-1}{\gamma}\right)\Delta s} = \rho^{\gamma}\,(1 + O(\epsilon^3)) \qquad (12a,b)$$

where γ is the ratio of specific heats, Δs is the difference in entropy $s - s^*$, made dimensionless with respect to the specific gas constant, $\bar{R}$, and where in Eq. (12b) use has been made of the fact that at most $\Delta s = O(\epsilon^3)$. Thus, from Eqs. (11) and (12b), one can derive the relations

$$P_1 = \gamma \rho_1 \qquad P_2 = \gamma \rho_2 + \frac{\gamma(\gamma-1)}{2} \rho_1^2 \qquad (13a,b)$$

When F(T) has been found, Eq. (10) can be used to obtain relations for the various approximations for the temperature in terms of the velocity perturbations. Finally, the equation of state

$$P = \rho \hat{T} \qquad (14)$$

may be used to give the necessary third set of equations.

The term F(T) may be evaluated using the energy equation in terms of the stagnation enthalpy. Thus, the rate of change of H along a particle path is

$$\frac{DH}{DT} = \frac{1}{\rho\gamma} \frac{\partial P}{\partial T} \qquad (15)$$

If H is expanded about its value far upstream of the duct, in the uniform flow, and the expansions for P and ρ are employed, then one finds from Eq. (15) that if

$$\tau = (k\epsilon^n)^{-1} , \qquad (16)$$

$$H = \frac{1}{2}(\gamma+1)/(\gamma-1) + \epsilon^{n+1} H_1 + \cdots , \qquad (17a)$$

$$\frac{\partial H_1}{\partial x} = \frac{k}{\gamma} \frac{\partial P_1}{\partial t} \qquad (17b)$$

Moreover, from the continuity equation

$$\frac{\partial \rho}{\partial T} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad (18)$$

one can show, in a similar manner, that $\partial(\rho_1 + u_1)/\partial x = 0$; since ρ and u are dimensionless with respect to their critical values in the incoming flow, and since the change in $\bar{a}^*$ across the shock wave due to unsteady motion is at most of order ϵ, then at the sonic surface ρ_1 and u_1 are both zero for all time, and the result therefore becomes

$$\rho_1 = -u_1 \tag{19}$$

Using Eqs. (19), (13a), and the first-order potential function, one can integrate Eq. (17b):

$$H_1 = -k\,\phi_{1_t} + g_1(y,t) \tag{20}$$

where g_1 is a function of integration. If Eqs. (20) and (17a) are substituted into the definition of H and the result along with Eqs. (8) and (16) is substituted into Eq. (10), one finds that

$$F(T) = \frac{\gamma+1}{2(\gamma-1)} + \epsilon^2 g_1 + \cdots \tag{21a}$$

$$a^2 = 1 + \frac{\gamma-1}{2}(1-q^2) + (\gamma-1)(H-H_\infty) \tag{21b}$$

and g_1 is a function of t alone. This is consistent with the result to be shown later that ϕ_1 is independent of y. The potential equation (9) can now be rewritten with $u = 1 + \phi_x$, $v = \phi_y$, and with a^2 given by Eq. (21b):

$$\begin{aligned}\phi_{yy} = {} & (\gamma+1)\phi_x\phi_{xx} + 2(1+\phi_x)\phi_{xT} + \phi_{TT} + (\gamma-1)\phi_x\phi_{yy} \\ & + \frac{1}{2}(\gamma+1)\phi_x^2\phi_{xx} - (\gamma-1)(H-H_\infty)(\phi_{xx}+\phi_{yy}) \\ & + O(\epsilon^{2+n}, \epsilon^4)\end{aligned} \tag{22}$$

The error estimate in Eq. (22) uses the anticipated results that $\phi_x = O(\epsilon)$, $\phi_y = O(\epsilon^2)$.

Ahead of the shock wave, following a streamline upstream to the undisturbed steady flow, one finds that $g_1 = 0$. It should be noted for future reference that there is no change expected in these results if an inner region enclosing the leading edges of the duct is traversed since no new oscillations are impressed upon the flow in this region.

In order to calculate g_1 downstream of the shock wave, the change in H across a moving wave must first be calculated. Since this change and other jump conditions are required for shock waves moving at velocities of varying order, it is necessary to consider the general jump conditions across a moving wave. These conditions are derived most conveniently in terms of a coordinate system normal and parallel to such a wave, at an arbitrary point on it, as shown in Fig. 2. There, $\vec{q}$ is the velocity, $\vec{n}$ and $\vec{t}$ are unit vectors in the directions normal and perpendicular to the wave, respectively, and c_n is the normal component of the velocity $\vec{c}$ of the shock wave at the point in question; the tangential component c_t is taken to be zero. If the superscript + is used to denote conditions relative to the shock wave, and double brackets [[ ]] indicate the jump in a quantity across the wave, then

$$[[H^+]] = [[h + \frac{(q_n^+)^2 + (q_t^+)^2}{2}]] = 0 \tag{23a}$$

$$q_n^+ = q_n - c_n \qquad\qquad q_t^+ = q_t \tag{23b,c}$$

Since $[[c_n]] = 0$ and $H = h + (q_n^2 + q_t^2)/2$, Eq. (23) can be written as

$$[[H]] = c_n [[q_n]] \tag{24}$$

and it remains to find c_n and q_n.

If the instantaneous shape of the shock wave is denoted by $x_s(y, T)$, and $S \equiv x - x_s$, then $S = 0$ on the shock wave and also

$$S_T + \vec{c} \cdot \nabla S = 0 \tag{25a}$$

$$\vec{n} = \frac{\nabla S}{|\nabla S|} = \frac{\vec{i} - x_{sy}\vec{j}}{\sqrt{1 + x_{sy}^2}} \tag{25b}$$

where $\vec{i}$ and $\vec{j}$ are unit vectors in the x and y directions, respectively, and the subscripts y and T denote partial derivatives. Using Eq. (25b) for ∇S in Eq. (25a) and for $\vec{n}$ in calculating q_n, one can show that

$$c_n = \frac{x_{sT}}{\sqrt{1 + x_{sy}^2}} \tag{26a}$$

$$q_n = \frac{u - v\,x_{sy}}{\sqrt{1 + x_{sy}^2}} \tag{26b}$$

Also, since $\vec{n} \times \vec{q} = \vec{n} \times q_t \vec{t}$,

$$q_t = \frac{v + u\,x_{sy}}{\sqrt{1 + x_{sy}^2}} \tag{27}$$

The shock-wave relations for one-dimensional flow are applicable in the direction normal to the wave, so that one can write immediately,

$$(q_n^+)_d \, (q_n^+)_u = (a^{+*})^2 \tag{28}$$

where subscripts u and d refer to conditions immediately upstream and downstream of the wave, respectively, and a^{+*} is the dimensionless critical sonic velocity in the coordinate system fixed to the wave. Hence,

$$\frac{(\gamma+1)}{2(\gamma-1)}(a^{+*})^2 = H^+ = H - c_n\left(q_n - \frac{c_n}{2}\right)$$

$$= H - \frac{1}{(1+x_{sy}^2)}\left(x_{sT}(u - v x_{sy}) - \frac{x_{sT}^2}{2}\right) \qquad (29)$$

In all of the problems considered here, $x_{sy} = O(\epsilon^{3/2})$, $v = O(\epsilon^2)$, $u = O(1)$, and $x_{sT} = O(\epsilon)$ at most. If the shock-wave velocity in the x direction is defined by $u_s = x_{sT}$, then up to and including terms $O(\epsilon^2)$ one can write:

$$c_n \sim x_{sT} \sim u_s \qquad (30a)$$

$$q_n \sim u \qquad\qquad q_t \sim u x_{sy} + v \qquad (30b,c)$$

$$(u - u_s)_d\,(u - u_s)_u \sim (a^{+*})^2 \qquad (30d)$$

$$\frac{(\gamma+1)}{2(\gamma-1)}(a^{+*})^2 \sim H - u_s u + \frac{u_s^2}{2} \qquad (30e)$$

It may be noted that x_s is desired only up to and including terms $O(\epsilon)$, which are independent of y.

If the inviscid flow field is considered to be that found in the limit as the viscosity tends to zero, such that the continuity of $\phi = \int_{-\infty}^{x} \phi_x\,dx$ + constant through the shock wave is retained, then ϕ and therefore ϕ_T are continuous and in fact unchanged across the shock wave. Hence, F(T) in Eq. (10) (and (21)) may be calculated for the flow downstream of the shock wave by calculating the jump in H across the shock from Eq. (24) and thus the jump in F(T) using Eq. (10) at the wave.

The boundary conditions to be used at each order of approximation may be found by noting that if $W \equiv y - y_w(x,t)$, then at the surface of the duct $W = 0$ and $W_T + \vec{q} \cdot \nabla W = 0$ there also. From this equation, one finds that

$$v_w = u_w \frac{\partial y_w}{\partial x} + \frac{1}{\tau} \frac{\partial y_w}{\partial t} \tag{31}$$

where the subscript w denotes conditions at the wall, or surface of the duct.

Finally, specific problems are defined by setting the relative orders of the parameters τ, α, and ϵ. First, we choose the orders of magnitude of α to be $\alpha = O(\epsilon^2)$ and $\alpha = O(\epsilon^3)$. That is, the amplitude of the wall motion is at most of the order of $y_w - 1$, which is equivalent to the thickness of the blades in a cascade. In general, the order of magnitude of τ should cover a range from $\tau \gg 1$ to $\tau = O(1)$. That is, for the case where the boundary layer apparently interacts with the moving shock wave the flow is quasi-steady so that $\tau \gg 1$; on the other hand, when the unsteadiness arises because of blade oscillation, the range for τ includes $\tau = O(1)$. For the problems to be considered here, we choose $\tau = O(\epsilon^{-2})$ and $\tau = O(\epsilon^{-1})$. To interpret these choices, we note that a weak pressure disturbance travels at a speed $|u - a|$, and it follows from Eqs. (5a), (21), and (8) that $u - a = O(\epsilon)$. Since the distance is O(1), physically the case $\tau = O(\epsilon^{-1})$ is that for which signals from the source of the disturbance reach the shock wave in a time comparable with the time characterizing the period of the disturbance, $\bar{T}_{ch}$, and the case

$\tau = O(\epsilon^{-2})$ is that for which signals reach the shock wave "instantaneously" compared to $\overline{T}_{ch}$, as shown in Ref. [15]. The case $\tau = O(1)$ is deferred to future work.

The specific problems considered here are then:

(1) Oscillations of Channel Walls with $\alpha = O(\epsilon^3)$ and of Back Pressure with Perturbations $O(\epsilon^2)$; $\tau = O(\epsilon^{-2})$.

In this case, because the walls are stationary to $O(\epsilon^2)$ and the pressure is steady to $O(\epsilon)$, unsteadiness in the flow velocity arises only in terms $O(\epsilon^2)$. Hence $u_s = O(\epsilon^2)$, and from Eq. (7b) it is seen, therefore, that $x_{s0} = x_{s0}(t)$; i.e., large-amplitude motion of the shock wave is possible. This is an extension of the problem considered in Ref. [12].

(2) Oscillations of Channel Walls with $\alpha = O(\epsilon^3)$ and of Back Pressure with Perturbations $O(\epsilon^2)$; $\tau = O(\epsilon^{-1})$

In this case, the flow is again unsteady in second order but the oscillations occur at a higher frequency than in problem (1), with a shock-wave motion of smaller amplitude; i.e., $u_s = O(\epsilon^2)$ again, so from Eq. (7b) x_{s0} = constant. Small-amplitude shock motion results.

(3) Oscillations of Channel Walls with $\alpha = O(\epsilon^2)$ and of Back Pressure with Perturbations $O(\epsilon)$; $\tau = O(\epsilon^{-1})$.

Here, the flow is unsteady in first order, and so $u_s = O(\epsilon)$. Hence, as in problem (1), $x_{s0} \neq$ constant so large-amplitude motion of the shock wave is possible.

Each of the problems listed here is applicable to a channel with an unsteady boundary layer and/or oscillating back pressure, and to blade flutter, although for the last application problem (1) has marginal use. In problems to be considered in the future, $\tau = O(1)$ with $\alpha = (\epsilon^3)$ in one case and $\alpha = O(\epsilon^2)$ in the other. These cases will be applicable to blade flutter with marginal application to unsteady boundary layers or oscillating back pressures because of the small-amplitude motion of the shock wave.

III. SOLUTIONS

In this section the three problems described in section II are considered in detail. In general, the goal has been to obtain solutions valid to order ϵ^2 in the velocity components and thus in the pressure, temperature, and density. As the problem complexity increases, this is not possible without the use of numerical methods of solution, and in those cases where considerable time would have been spent in obtaining these higher-order corrections, the problem formulations for these terms have been completed, but no solutions are given.

Problem (1): $\alpha = O(\epsilon^3)$, $\tau = O(\epsilon^{-2})$

There are several regions in which the main channel-flow solutions do not describe the flow properly; i.e., they are not uniformly valid there. In that event, special "inner" solutions which hold in the regions in question and which match with the "outer" channel-flow solutions must be derived. The inner and outer solutions may then be joined to form composite solutions valid to the desired order of magnitude in larger regions. For the problems considered here, inner solutions are needed at the entrance to the duct (and perhaps at the exit), at the throat, and in the neighborhood of the shock wave. In addition, a separate calculation must be made for the velocity and position of the shock wave. Because the analyses are similar in all of the problems to be considered, each of these special calculations is contained in a titled subsection for easy reference.

In this section, the equations used to relate α to ϵ and τ to ϵ are

$$\alpha = \epsilon^3 \tag{32a}$$

$$\tau = (k\,\epsilon^2)^{-1} \tag{32b}$$

where k is an arbitrary constant of order one.

Channel-Flow Solutions

In order to find the governing equations for each order of approximation from Eq. (9), it is necessary first to calculate a, the speed of sound. As shown previously, $F(T) = \frac{1}{2}(\gamma + 1)/(\gamma - 1)$ upstream of the shock wave and so a^2 may be found from Eq. (10) immediately. Downstream of the shock wave the jump in H is found using Eqs. (24), (30a) and (30b), with $u_s = k\epsilon^2\, dx_{s0}/dt + \cdots$ from Eq. (7b); thus, one can show that H_d and g_1, defined in Eq. (21), are

$$H_d - H_u = \epsilon^3\, k\, \frac{dx_{s0}}{dt}\,[u_{1d} - u_{1u}] \tag{33a}$$

$$g_1 = 0 \tag{33b}$$

Since only terms $O(\epsilon^3)$ are needed in Eq. (9), only terms up to and including $O(\epsilon^2)$ need be retained in a^2, and so $F(T) = \frac{1}{2}(\gamma + 1)/(\gamma - 1)$ suffices throughout the channel in this case.

If Eq. (10) is substituted into Eq. (9), and Eq. (8) substituted into the resulting equation, one finds the following governing equations for the ϕ_i by gathering terms of order ϵ, ϵ^2, and ϵ^3, respectively, and rearranging slightly:

$$\phi_{1yy} = 0 \tag{34a}$$

$$\phi_{2yy} = \left[\left(\frac{\gamma+1}{2}\right) \phi_{1x}^2 \right]_x \tag{34b}$$

$$\phi_{3yy} = \left[(\gamma+1)\, \phi_{1x} \phi_{2x} + 2k\phi_{1t} + \frac{1}{3} (\gamma - \frac{1}{2}) (\gamma+1) \phi_{1x}^3 \right]_x \tag{34c}$$

where subscripts x, y, and t denote partial derivatives. If Eqs. (5), with of course $u = 1 + \phi_x$ and $v = \phi_y$, Eqs. (4), and Eqs. (32) are substituted into Eq. (31), giving the boundary conditions, one finds that at the upper and lower walls, respectively,

$$\phi_{2y}(x, 1, t) = -f_u' \quad , \quad \phi_{3y}(x, 1, t) = G_{u_x} - u_1 f_u' \tag{35a,b}$$

$$\phi_{2y}(x, -1, t) = f_\ell' \quad , \quad \phi_{3y}(x, -1, t) = -(G_{\ell_x} - u_1 f_\ell') \tag{35c,d}$$

where $f_u' = df_u/dx$, etc.

From Eq. (34a), since there is no term $O(\epsilon)$ in the boundary conditions,

$$\phi_1 = \phi_1(x, t) \tag{36}$$

Because ϕ_1 is independent of y, Eq. (34b) may be integrated easily. Upon satisfying the boundary conditions on ϕ_{2y}, given by Eqns. (35a,c), one finds an equation involving only ϕ_{ix}, f_u, and f_ℓ which, although nonlinear, again is easily integrated to give:

$$\phi_{1x} = \pm \sqrt{\frac{2}{(\gamma+1)} \quad (c_1 - f)} \tag{37a}$$

$$f \equiv \frac{f_u + f_\ell}{2} \tag{37b}$$

where, in general, $c_1 = c_1(t)$. Now, c_1 is set by flow conditions at the throat, the duct exit, or the duct entrance. For example, if the flow is choked, $\phi_{1x} = 0$ at the throat. On the other hand, if the flow throughout the duct is subsonic, conditions at the exit set c_1, and if the entrance flow is supersonic, conditions at the entrance are used. In the case of primary interest here, the entrance flow is subsonic and choked at the throat with a shock wave downstream of the throat. It will be shown later that because f is independent of time and only second-order pressure oscillations are considered, c_1 is independent of time for this problem. Hence, if we set $x = 0$ at the point where $f' = 0$,

$$c_1 = f(0) \tag{38}$$

Moreover, $\phi_{1t} = 0$, and using Eq. (37a) for ϕ_{1x}, the solution for ϕ_2 may be written simply as

$$\phi_2 = - f' \frac{y^2}{2} + g_2 y + h_2(x,t) \tag{39a}$$

$$g_2 = \frac{(f'_\ell - f'_u)}{2} \tag{39b}$$

where h_2 is a function of integration.

With ϕ_1 and ϕ_2 known, Eq. (34c) may be integrated once to give ϕ_{3y}. If the boundary conditions are applied at $y = \pm 1$ (Eqs. (35b,d)) and the resulting equations are subtracted, a governing equation for $h_2(x,t)$ is obtained. This equation may be integrated once to give h_{2x}, which is all that is needed for the velocity components. The result is

$$h_{2x} = \frac{1}{6}\left[f'' - (2\gamma - 3)\,\phi_{1x}^2\right] + \frac{c_2(t) + G}{(\gamma+1)\,\phi_{1x}} \tag{40a}$$

$$G = \frac{G_u + G_\ell}{2} \tag{40b}$$

In Eq. (40a), $c_2(t)$ is a function of integration which is set by given flow conditions at some station in the duct; in general it has different values upstream and downstream of the shock wave.

With these results, the solutions for the velocity components may be written up to and including terms $O(\epsilon^2)$:

$$u = 1 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

$$= 1 \pm \epsilon \sqrt{\frac{2}{(\gamma+1)}(c_1 - f)} + \epsilon^2\left[- f''\frac{y^2}{2} + g_2' y + h_{2x}\right] + \cdots \tag{41a}$$

$$v = \epsilon^2 v_2 + \cdots = \epsilon^2 \left[-f' y + g_2\right] + \cdots \tag{41b}$$

If the expansions for u and v are substituted into Eq. (10) with $F(T) = \frac{1}{2}(\gamma+1)/(\gamma-1)$, an expansion for $T = a^2$ results; this, with Eqs. (13a,b), (14), and (19), allows the following relations to be derived for the density, temperature, and pressure:

$$\rho = 1 - \epsilon u_1 - \epsilon^2 \left(u_2 + \left(\frac{\gamma-1}{2}\right) u_1^2\right) + \cdots \tag{42a}$$

$$\hat{T} = 1 - \epsilon(\gamma - 1)\, u_1 - \epsilon^2 (\gamma-1)\left(u_2 + \frac{u_1^2}{2}\right) + \cdots \tag{42b}$$

$$P = 1 - \epsilon \gamma u_1 - \epsilon^2 \gamma u_2 + \cdots \tag{42c}$$

Thus, the solutions found may be used to calculate velocity, density, temperature and pressure distributions across the channel at a given location in the flow or along the centerline or on the wall. They are valid in the main part of the channel.

It is seen from Eqs. (42c), (41a), and (40a), that imposing a given pressure oscillation of order ϵ^2 at the exit of the duct, say, where $x = x_1$, is equivalent to prescribing $c_2 + G$ there. Thus if we take $f = f' = f'' = 0$ at $x = x_1$, and further set

$$-\frac{P_2(x_1,t)}{\gamma} = u_2(x_1,t) = -\frac{(2\gamma-3)}{6}u_1^2(x_1) + \frac{c_{2d}^{(s)} + G_d(t)}{(\gamma+1)u_1(x_1)} \tag{43}$$

then the last term describes the oscillations in back pressure imposed on the flow. The constant $c_{2d}^{(s)}$ refers to a steady-state back pressure, giving a steady-state location for the shock wave, and $G_d(t)$ refers to the unsteady fluctuations in back pressure superimposed over the steady-state value. Then, downstream of the shock wave, $c_2(t)$ is set by the following equation:

$$c_2(t) + G(x_1,t) = c_{2d}^{(s)} + G_d(t) \tag{44}$$

Upstream of the shock wave, if the flow is choked,

$$c_2(t) + G(0,t) = 0 \tag{45}$$

if h_{2x} is to remain finite there (see Eqn. (40a)). This is discussed in greater detail later, and it is also shown that the solution for u can be written in composite form such that the singular behavior of h_{2x} at the throat does not arise.

It is important to note that the oscillations in back pressure were chosen here to give flow fluctuations of the same order, namely $O(\epsilon^2)$, as those induced by the oscillations of the wall for $\alpha = O(\epsilon^3)$. This was done so that relevant comparisons of the two effects could be made.

Leading-Edge Solution

Near the leading edge $x = x_0$ the "channel" solutions derived above are no longer correct because the flow is no longer nearly one-dimensional. A direct matching of the channel solutions with the corresponding undisturbed-flow quantities in general is not possible. This was demonstrated in Ref. [16] for steady flow at high subsonic speed through a cascade with staggered blades having wedge-shaped leading edges, with a simple correction included for rounded edges. To illustrate the need for an inner leading-edge solution in the present context, the simpler case of symmetric cusped edges is chosen, so that $f_u'(x_0) = f_\ell'(x_0) = 0$ and $f_u''(x_0) = f_\ell''(x_0) = f''(x_1) > 0$; wall motions are such that $G_u(x_0,t) = G_\ell(x_0,t) = G(x_0,t)$; also $f_u(x_0) = f_\ell(x_0) = 0$ by definition. The leading-edge solution is found to be necessary for providing correct flow details near the edges, but is not needed, at least in the present applications, for determination of the integration constants c_1 and c_2 in the channel solution. Solutions for wedge-shaped edges would be somewhat more complicated, and would lead to the same general conclusion.

For symmetric cusped leading edges, expansion of the solutions (41a,b) for u and v as $x \rightarrow x_0$ gives, for the subsonic case where

$\phi_1'(x) < 0$ in the forward part of the duct,

$$u = 1 - \epsilon \left(\frac{2\, c_1}{\gamma + 1} \right)^{1/2} \left\{ 1 - \frac{f''(x_0)}{4\, c_1} (x - x_0)^2 + \cdots \right\}$$

$$+ \epsilon^2 \left\{ \frac{f_0''}{2} \left(\frac{1}{3} - y^2\right) + \left(\frac{1}{2} - \frac{\gamma}{3} \right) \frac{2\, c_1}{\gamma + 1} - \frac{c_2(t) + G(x_0, t)}{[2\,(\gamma + 1)\, c_1]^{1/2}} + \cdots \right\}$$

$$+ \cdots \qquad (46a)$$

$$v = -\epsilon^2 f''(x_0)\,(x - x_0)\, y + \cdots \qquad (46b)$$

The undisturbed-flow velocity is represented (Fig. 1) as a series expansion in ϵ by

$$u_\infty = 1 + \epsilon\, u_{1\infty} + \epsilon^2\, u_{2\infty} + \cdots \qquad (47)$$

where $u_{1\infty} < 0$ for the subsonic case. The velocity perturbation $u - 1 = \phi_x$ is also $O(\epsilon)$ in the leading-edge region, and the term $(\gamma + 1)\, \phi_x\, \phi_{xx}$ in the differential equation (22) is no longer small in comparison with ϕ_{yy} when $x - x_0 = O(\epsilon^{1/2})$, corresponding to a thin vertical strip containing the edges. For flows of greatest practical interest, appreciable velocity changes in a small leading-edge region are probably undesirable, and so we consider only cases in which the $O(\epsilon)$ term is constant there. Comparison of equations (46a) and (47) then allows c_1 to be related to $u_{1\infty}$, which is considered to be given:

$$c_1 = \frac{\gamma + 1}{2}\, u_{1\infty}^2 \qquad (48)$$

The leading-edge flow depends on coordinates x^* and y^*, defined by

$$x^* = \frac{x - x_0}{(K\epsilon)^{1/2}} \quad , \quad y^* = y \tag{49}$$

where the factor $K^{1/2}$ is included for convenience, with K defined by

$$1 - M_\infty^2 = K\epsilon + O(\epsilon^2) = -(\gamma + 1)\, u_{1\infty}\, \epsilon + O(\epsilon^2) \tag{50}$$

The perturbation potential has the form

$$\phi = (u_\infty - 1)\, x + \epsilon^{5/2}\, \phi_2^*\, (x^*, y^*, t) + \cdots \tag{51}$$

A complex velocity can therefore be written as

$$u - i\, \frac{v}{(\epsilon K)^{1/2}} = u_\infty + \epsilon^2\, K^{-1/2}\, (\phi^*_{2x^*} - i\, \phi^*_{2y^*}) + \cdots \tag{52}$$

Retaining the largest terms in the differential equation (22) gives Laplace's equation

$$\phi^*_{2x^*x^*} + \phi^*_{2y^*y^*} = 0 \tag{53}$$

The wall boundary conditions are

$$\phi^*_{2y^*}(x^*, \pm 1, t) = \mp\, K^{1/2}\, x^*\, f''(x_0), \qquad 0 < x^* < \infty \tag{54}$$

for the interior walls, and $\phi^*_{2y^*}(x^*, \pm 1, t) = 0$ for $0 < x^* < \infty$ along the exterior surfaces; these boundary conditions are indicated in Fig. 3a. The leading-edge solution for u must match with the channel solution for u; that is, the expansion of the leading-edge solution as $x^* \to \infty$, for

$-1 < y^* < 1$, is required to agree term by term with the expansion of the channel solution as $x \to x_0$, shown in Eq. (46a). Finally, as $x^{*2} + y^{*2} \to \infty$ outside the channel, the velocity perturbations must approach zero.

The solution to the flow problem formulated above can be found with the help of a conformal transformation from the complex z^*-plane, where $z^* = x^* + iy^*$, to a complex ζ-plane. The boundaries shown in Fig. 3a are therefore transformed to the real axis in Fig. 3b, where the leading edges $z^* = \pm i$ have been mapped onto the points $\zeta = \pm 1$. The required transformation is

$$z^* = \frac{1}{\pi}(\zeta^2 - 1) - \frac{2}{\pi}\ln\zeta + i \tag{55}$$

If the complex potential corresponding to ϕ_2^* is $F(z^*) = \mathfrak{F}(\zeta)$, the complex velocity is

$$F'(z^*) = \mathfrak{F}'(\zeta)\, d\zeta/dz^* = \phi^*_{2x^*} - i\,\phi^*_{2y^*} \tag{56}$$

A simpler preliminary boundary-value problem is obtained by differentiation with respect to x^*, since the wall values of $\phi^*_{2x^*y^*}$ are constant. The solution to this problem is

$$F''(z^*) = \phi^*_{2x^*x^*} - i\,\phi^*_{2x^*y^*} = \frac{K^{1/2} f''(x_0)}{\pi}\ln\left(1 - \frac{1}{\zeta^2}\right) \tag{57}$$

Integrating, and adding a source term at the origin $\zeta = 0$ so that the matching condition with the channel solution can be satisfied, one finds finally

$$\phi^*_{2x^*} - i\,\phi^*_{2y^*} = \frac{K^{1/2} f''(x_0)}{\pi^2}\left\{1 + (\zeta^2 - 1)\ln\left(1 - \frac{1}{\zeta^2}\right)\right.$$

$$\left. + 2\int_\zeta^\infty \frac{1}{\zeta}\ln\left(1 - \frac{1}{\zeta^2}\right)d\zeta\right\} + \frac{K^{1/2}}{2}\left\{\frac{1}{\pi^2} f''(x_0) + m_2(t)\right\}\frac{1}{\zeta^2 - 1} \tag{58}$$

where $m_2(t)$ is still to be determined.

As $z^* \to \infty$ outside the channel (i.e., with arg z^* fixed, $0 < \arg z^* < 2\pi$), the transformation (55) becomes $z^* = \zeta^*/\pi + \cdots$, and expansion of $F'(z^*)$ gives velocity perturbations

$$u - u_\infty - i\,\frac{v}{(\epsilon K)^{1/2}} = \epsilon^2\,\frac{m_2(t)}{2\pi z^*} + \cdots \tag{59}$$

This expression describes the velocity due to a source of strength $m_2(t)$ at $z^* = 0$, and provides a matching condition for velocity perturbations in the external flow, which could be calculated if desired. For such an outer solution the proper length scale in the flow direction would be the duct length, so that $x = O(1)$. In the differential equation the terms ϕ_{yy} and $(\gamma+1)\,\phi_x\,\phi_{xx} \sim (\gamma+1)\,\epsilon\,u_{1\infty}\,\phi_{xx}$ are then of the same order (and all other terms are of higher order) when $y = O(\epsilon^{-1/2})$, so that an appropriate y-coordinate would be $\tilde{y} = (K\epsilon)^{1/2}y$. For $x = O(1)$ and $\tilde{y} = O(1)$ the largest perturbations on the uniform flow are described by a velocity potential which satisfies Laplace's equation in x and $\tilde{y}$. For matching with the leading-edge result, the solution would contain one term which represents a source at $x = 0$, $\tilde{y} = 0$; the remaining part of the solutions would be determined by the prescribed unsteady normal velocity component at the (exterior) duct walls located, to the outer length scale, at $\tilde{y} = 0$. The details of such outer solutions are not, however, of special interest here.

At the edges $z^* = \pm\, i$, the leading-edge solution for the velocity has the inverse-square-root singularity associated with flow around a sharp edge. As $z^* \rightarrow \pm\, i$,

$$u - u_\infty - \frac{iv}{(\epsilon K)^{1/2}} \sim \pm\, \epsilon^2 \, \frac{m_2(t) + f''(x_0)/\pi^2}{2(2\pi\, |z^* \mp i|)^{1/2}} \exp\{-\tfrac{i}{2} \arg(z^* \mp i)\} \quad (60)$$

A positive or negative value of $m_2(t) + f''(x_0)/\pi^2$ implies an outflow or an inflow, respectively, around the edges. An outflow around the edges remains even if the source strength $m_2(t)$ is zero, since the flow ahead of the duct anticipates the area decrease and accelerates slightly before reaching the duct. Thus a slight contraction of streamtube area occurs upstream of the duct, and the small excess mass flow spills around the edges.

The function $m_2(t)$ is found from the matching of the leading-edge and channel solutions. As $x^* \rightarrow \infty$ with $-1 < y^* < 1$, the transformation (56) implies that $\zeta \rightarrow 0$, and the expansion as $\zeta \rightarrow 0$ of the solution (58) can be shown to give

$$u - u_\infty = \frac{1}{2}\, \epsilon^2 \, \{f''(x_0)\,(x^{*2} - y^{*2} + \tfrac{1}{3}) - m_2(t)\} + \cdots \quad (61a)$$

$$v = -\, \epsilon^{5/2}\, K^{1/2}\, x^*\, y^*\, f''(x_0) + \cdots \quad (61b)$$

Comparison of Eqs. (61a) and (46a) shows that

$$m_2(t) = 2u_{2\infty} - (1 - \frac{2}{3}\gamma)\, u_{1\infty}^2 - 2\,\frac{c_2(t) + G(x_0,t)}{(\gamma+1)\, u_{1\infty}} \qquad (62)$$

Here $u_{1\infty}$, $u_{2\infty}$, and $G(x_0,t)$ are given quantities, and $c_2(t)$ is also considered to be known. For a purely subsonic flow, $c_2(t)$ is determined by the imposed pressure downstream of the duct, as shown in Eq. (44) for cases in which no special trailing-edge solution is needed. For example, the duct might have zero slope and curvature at trailing edges which are joined to fixed parallel walls downstream; or the flow might exhaust into a plenum at $x = x_1$, again with $f_u''(x_1) = f_\ell''(x_1) = 0$ and with the flow leaving the edges smoothly, i.e., with the wall streamlines having continuous curvature. Of primary interest here, however, is the case of choked flow, for which $c_2(t)$ is determined by conditions at the throat, as anticipated in Eq. (45) and as discussed below.

Flow Details Near the Throat

At a throat, the flow velocity may have a maximum (for subsonic flow) or a minimum (for supersonic flow), or the flow may accelerate or decelerate through the sound speed. In any case, if $|u - 1| \ll \epsilon$ near a throat at $x = 0$, then $\phi_{1x} \to 0$ as $x \to 0$ and, from Eq. (40a), $h_{2x} \to \infty$ as $x \to 0$. Since ϕ_{2x} therefore does not remain small in comparison with ϕ_{1x} near $x = 0$, it should be anticipated that the channel-flow formulation may not always be correct near a throat, and that for some combinations of the parameters a local "inner" solution in a thin throat region might be needed. In order to understand what modifications might be required, one might consider a purely subsonic flow with the back pressure gradually decreasing (or the undisturbed-flow Mach number increasing) until sonic speed is reached in the throat region. This approach is taken in the following paragraphs, and two special cases are seen to arise, corresponding to $u - 1 = O(\epsilon^{3/2})$ and $u - 1 = O(\epsilon^2)$ near the throat. In the latter case, acceleration to supersonic speed is considered in detail.

According to the channel solution (41a), the (subsonic) velocity near a throat located at $x = 0$ is

$$u = 1 - \epsilon \left(\frac{2}{\gamma+1}\right)^{1/2} \left\{c_1 - f(0) - \frac{1}{2} x^2 f''(0) + \cdots\right\}^{1/2} + \epsilon^2 \left\{\frac{1}{2} f''(0) \left(\frac{1}{3} - y^2\right) + \frac{1}{2}[f_\ell''(0) - f_u''(0)] y + \left(\frac{1}{2} - \frac{\gamma}{3}\right) \frac{2}{\gamma+1} [c_1 - f(0)] - \frac{G(0,t) + c_2(t)}{[2(\gamma+1)]^{1/2} [c_1 - f(0) - \frac{1}{2} x^2 f''(0) + \cdots]^{1/2}}\right\} + \cdots \qquad (63)$$

where $c_1 = (\gamma+1) u_{1\infty}^2/2$, from Eq. (48). If the flow is choked, then $c_1 = f(0)$; i.e., $(\gamma+1) u_{1\infty}^2/2 = f(0)$. The second term in Eq. (63) is then no longer small in comparison with the first term when $x = O(\epsilon^{1/2})$ and $u - 1 = O(\epsilon^{3/2})$. If a special local solution corresponding to these orders is assumed necessary, one finds that the differential equation and boundary conditions lead to a problem formulation for $x = O(\epsilon^{1/2})$ which is similar in form to that which leads to the solution (37a) for

ϕ_{1x}. The local solution is found to give

$$u = 1 - \epsilon^{3/2} \left(\frac{2}{\gamma+1}\right)^{1/2} \left\{ -\frac{1}{2}\frac{x^2}{\epsilon} f''(0) + c_2(t) + G(0,t) \right\}^{1/2} + \cdots \tag{64}$$

where the functions of t are chosen for matching as $x/\epsilon^{1/2} \to -\infty$ with the expansion (63) of the channel solution as $x \to 0$. It can be seen that $c_2(t) + G(0,t) > 0$ for subsonic flow at the throat. The result (64) remains correct at $x = 0$ and simply describes a change in fluid acceleration from positive to negative values within a short distance $x = O(\epsilon^{1/2})$ of the throat. A convenient composite solution, suggested in Ref. [17] for steady flow, is obtained by replacing the integration constant c_1 with a suitable series expansion in ϵ :

$$u = 1 - \epsilon \left(\frac{2}{\gamma+1}\right)^{1/2} \{c_1 - f(x) + \epsilon\, c_2(t) + \epsilon\, G(x,t) + \cdots\}^{1/2}$$
$$+ \epsilon^2 \{ \tfrac{1}{2} f''(x) \left(\tfrac{1}{3} - y^2\right) + \tfrac{1}{2} [f_\ell''(x) - f_u''(x)]\, y + \left(\tfrac{1}{2} - \tfrac{\gamma}{3}\right) \frac{2}{\gamma+1} [c_1 - f(x)] \}$$
$$+ \cdots \tag{65}$$

The solution (41a) is recovered if the square root is expanded for $|\epsilon\, c_2(t) + \epsilon\, G(x,t)| << c_1 - f(x)$, and if $c_1 = f(0)$, the local solution (64) is obtained when $x = O(\epsilon^{1/2})$.

If now $c_1 - f(0) = 0$ and also $c_2(t) + G(0,t) = 0$, either the channel solution or the composite solution implies that no difficulty now arises when $x = O(\epsilon^{1/2})$, but that when $x = O(\epsilon)$ the largest term in $u - 1$ is $O(\epsilon^2)$ and is a function of y as well as x/ϵ. This conclusion also follows from consideration of the differential equation and boundary conditions.

The wall boundary condition (31) shows that $\phi_y = O(\epsilon^2 x)$ as $x \to 0$. In the potential equation (22), it is expected that $(\gamma + 1)\, \phi_x\, \phi_{xx}$ is no longer small in comparison with ϕ_{yy} when the local Mach number is sufficiently close to one. Since also $y = O(1)$, it follows from equating the orders of these terms that a local solution satisfying a nonlinear differential equation near the throat may be needed when $x = O(\epsilon)$. Coordinates x^* and y^* are defined by

$$x^* = \frac{x}{(\gamma+1)^{1/2}\,[-f''(0)]^{1/2}\epsilon} \quad , \qquad y^* = y \tag{66}$$

and the perturbation potential is represented in the form

$$\phi = (\gamma+1)^{1/2}\,[-f''(0)]^{3/2}\,\epsilon^3\,\phi_2^*(x^*, y^*, t) + \cdots \tag{67}$$

The largest terms in the differential equation and boundary conditions give

$$\phi^*_{2y^*y^*} = \phi_{2x^*}\,\phi^*_{2x^*x^*} \tag{68a}$$

$$\begin{aligned}\phi^*_{2y^*}(x^*, \pm 1, t) = &\left\{\pm 1 - \frac{f_\ell''(0) - f_u''(0)}{2\, f''(0)}\right\} x^* \\ &+ \left\{\mp 1 + \frac{G_{\ell x}(0,t) - G_{ux}(0,t)}{2\, G_x(0,t)}\right\} x_1^*\end{aligned} \tag{68b}$$

where

$$x_1^* = -\,(\gamma+1)^{-1/2}[-f''(0)]^{-3/2}\, G_x(0, \tag{69}$$

Of particular interest is the solution which describes acceleration of the flow from subsonic to supersonic speeds. As for steady flow, ϕ_2^*

is found as a polynomial in x^* and y^*, but now some time-dependent terms appear:

$$\phi_2^* = \frac{1}{2}(x^* - x_1^*)^2 + \frac{1}{2}(x^* - x_1^*)(y^{*2} - \frac{1}{3}) + \frac{1}{12}(\frac{1}{2}y^{*4} - y^{*2})$$
$$- \frac{1}{2}\{f_\ell''(0) - f_u''(0)\}\{x^* y^* + \frac{1}{2}(\frac{1}{3}y^{*3} - y^*)\}/f''(0)$$
$$- \frac{1}{2}\{G_{\ell x}(0,t) + G_{ux}(0,t)\}\; x_1^* y^*/G_x(0,t) \qquad (70)$$

In the original variables the velocity components for $x = O(\epsilon)$ are then

$$u = 1 + \epsilon^2 \left\{ \frac{x[-f''(0)]^{1/2}}{(\gamma+1)^{1/2}\epsilon} + \frac{G_x(0,t)}{(\gamma+1)^{1/2}[-f''(0)]^{1/2}} - \frac{1}{2}f''(0)(y^2 - \frac{1}{3}) \right.$$
$$\left. + \frac{1}{2}[f_\ell''(0) - f_u''(0)]\, y \right\} + \cdots \qquad (71a)$$

$$v = \epsilon^3 \{- f''(0)\frac{x}{\epsilon}y + G_x(0,t)y + \frac{1}{6}(\gamma+1)^{1/2}[-f''(0)]^{3/2}(y^3 - y)$$
$$+ \frac{1}{2}[f_\ell''(0) - f_u''(0)][\frac{x}{\epsilon} + \frac{1}{2}(\gamma+1)^{1/2}[-f''(0)]^{1/2}(y^2 - 1)]$$
$$+ \frac{1}{2}[G_{\ell x}(0,t) + G_{ux}(0,t)]\} + \cdots \qquad (71b)$$

The sonic line, defined by $u = 1$, is located at

$$x = \epsilon \left\{ \frac{G_x(0,t)}{f''(0)} + \frac{1}{2}(\gamma+1)^{1/2}[-f''(0)]^{1/2}(\frac{1}{3} - y^2) \right.$$
$$\left. - (\gamma+1)^{1/2}\frac{f_\ell''(0) - f_u''(0)}{2[-f''(0)]^{1/2}}\, y \right\} + \cdots \qquad (72)$$

The second and third terms give the steady-state result, and the first term is seen to be the value of x at which the instantaneous channel width is a minimum. That is, the sonic line is found to have the appropriate quasi-steady location at each instant. This conclusion follows only because of the particular choices made here for the orders of magnitude of τ and α. For the other problems discussed in subsequent sections, the sonic line does not remain close to a quasi-steady location.

The expansion (63) of the channel solution as $x \to 0$ can now be matched with the throat solution (71a) to give the results anticipated earlier in Eqs. (38) and (45):

$$c_1 = f(0)\ , \qquad c_2(t) = -\,G(0,t) \qquad \text{(73a,b)}$$

Also $c_1 = \frac{1}{2}(\gamma+1)\,u_{1\infty}^2$, from Eq. (48), and so $u_{1\infty} = -\,2^{1/2} f(0)/(\gamma+1)^{1/2}$. That is, at the minimum Mach number for choking

$$u_\infty = 1 - \epsilon\, 2^{1/2} f(0)/(\gamma+1)^{1/2} + O(\epsilon^2) \qquad (74)$$

The function $c_2(t)$ is found here from consideration of the flow details near $x = 0$, and the result of course agrees with Eq. (45), which was obtained from the requirement that ϕ_{2x} remain finite as $x \to 0$. Comparison of the channel and throat solutions (41) and (71) shows that, for acceleration from subsonic to supersonic speeds, the channel solutions for u and v do in fact remain correct to order ϵ^2 in the thin region $x = O(\epsilon)$ which includes the throat. A special local solution would be needed here only for flows which remain either subsonic or supersonic when $x = O(\epsilon)$, but which approach so close to the sound speed that the minimum (nonzero) value of $|u - 1|$ is $O(\epsilon^2)$. In the present case, the ratio $\{c_2(t) + G(x,t)\}/\phi_{1x}$ is, however, indeterminate at $x = 0$ and must of course be evaluated properly there. The combination $c_1 + \epsilon c_2$ appearing in the composite solution (64) is equal to $f(0) - \epsilon G(0,t)$, which is just the (relative) area change at $x = 0$. This earlier composite solution can be modified to include the case $c_2(t) = - G(0,t)$ by addition of a suitable $O(\epsilon^2)$ term inside the square root:

$$u = 1 \mp \epsilon \left(\frac{2}{\gamma+1}\right)^{1/2} \left\{ c_1 - f(x) + \epsilon c_2(t) + \epsilon G(x,t) + \frac{\epsilon^2 G_x^2(0,t)}{2[-f''(0)]} \right\}^{1/2}$$

$$+ \epsilon^2 \left\{ \frac{1}{2} f''(x) \left(\frac{1}{3} - y^2\right) + \frac{1}{2}[f_\ell''(x) - f_u''(x)]\, y + \left(\frac{1}{2} - \frac{\gamma}{3}\right) \frac{2}{\gamma+1} [c_1 - f(x)] \right\}$$

$$+ \ldots \qquad (75)$$

This new composite solution is correct to order ϵ^2 both near and away from the throat, since it may be expanded for $x = O(\epsilon)$ to give the throat solution and for $x = O(1)$ to give the channel solution. Moreover, the case $c_2(t) + G(0,t) > 0$ is also included, since then Eq. (64) is recovered by expansion of Eq. (75) for $x = O(\epsilon^{1/2})$. Thus Eq. (75) is sufficient for purely subsonic flow, with $c_2(t) + G(0,t) > 0$, or for acceleration from subsonic to supersonic speed, with $c_2(t) + G(0,t) = 0$, and special considerations are no longer necessary for $x = O(\epsilon^{1/2})$ or $x = O(\epsilon)$.

Velocity and Position of the Shock Wave

Just as in the case where the back pressure alone oscillates [9-12], the position of the shock wave at any instant is found by deriving first the expression for the instantaneous velocity of the shock wave and integrating this equation. The velocity is found by satisfying the jump conditions across the shock wave. It was shown earlier that the flow is steady in first order. If the velocity of the shock wave were $O(\epsilon)$, then unsteadiness would be induced in the $O(\epsilon)$ velocity terms downstream of the shock wave, so it is clear that for the conditions chosen $u_s = O(\epsilon^2)$; this is consistent with the order of u_s found using Eqs. (7b) and (32b).

If the expansion for u given in Eq. (5a) and the value $H = (\gamma + 1)/2(\gamma - 1)$ are substituted into Eqs. (30c,d), it is seen that the jump conditions across the shock wave are

$$u_{1d} = - u_{1u} \tag{76a}$$

$$u_{2d} = - u_{2u} - u_{1d}\, u_{1u} + \frac{4}{(\gamma + 1)} \left(\frac{u_s}{\epsilon^2}\right) \tag{76b}$$

It is clear from Eqs. (41a) that if the solution for u_1 with the positive sign (supersonic flow) is associated with the flow upstream of the shock wave and that with the negative sign (subsonic flow) is associated with the flow downstream of the shock wave, the jump conditions on u_1 are satisfied. On the other hand, because of the dependence of u_2 on y, it is clear also that the second-order jump conditions cannot be satisfied by the channel-flow solutions; additional calculations are required.

Because the second-order jump conditions are not satisfied, it is necessary to consider a thin inner region behind the shock wave, within which some adjustment of the flow must take place. The solutions in this inner region, which turns out to have an extent $\Delta x = O(\epsilon^{1/2})$, must satisfy the jump conditions across the shock wave and the boundary conditions at the walls, and must match with the outer channel flow solutions in the appropriate limit. The problem here is complicated by the fact that the shock wave, and therefore the inner region, is moving such that $\Delta x_s = O(1)$. Solutions for such an inner region enclosing the shock wave were first found for steady flow [17]. This solution was extended [9] to cover unsteady flow with small-amplitude motion of the shock wave, with $\Delta x_s = O(\epsilon)$ and $\tau = O(\epsilon^{-1})$, so that $u_s = O(\epsilon^2)$. In this analysis second-order oscillations in the back pressure caused the unsteadiness in the flow, the walls being stationary. Next, it was shown in Refs. [15] and [12] that for impressed pressure fluctuations of the same order, but with $\tau = O(\epsilon^{-2})$ so that $\Delta x_s = O(1)$ and $u_s = O(\epsilon^2)$, the same form of solution resulted in the inner region. Because no time derivatives remain in the governing equations for this region, to the order considered, the differences in the solution arise only from the differences in the matching conditions to be met; for $\tau = O(\epsilon^{-1})$ the shock is stationary and the incoming velocities evaluated at the shock are constant to lowest order while for $\tau = O(\epsilon^{-2})$, the position of the wave and the incoming velocities are functions of time to lowest order. As a result, the equations for the velocity of the shock wave have the same dependence upon h_{2x}, f, etc. in the two cases, but in each case these functions depend differently upon time. Also $u_s = \epsilon^2 \, dx_{s1}/dt$ for $\tau = O(\epsilon^{-1})$, whereas $u_s = \epsilon^2 \, dx_{s0}/dt$ for $\tau = O(\epsilon^{-2})$, as seen from Eq. (7b).

The analyses referred to in Refs. [17, 9, 15, 12] all involved symmetric channels. However, Richey [18] considered the case of asymmetric walls also, for $\tau = O(\epsilon^{-1})$, and showed that a simple transformation of the y variable in the inner region could be used to relate the solutions for symmetric and asymmetric walls. His outer channel-flow solutions have the same form as those found here to second order, even though τ is of different order in each problem, because in both cases the flow is steady in first order and the walls are stationary to second order. In view of the above remarks, it is clear that in the inner region behind the shock wave, those solutions found before, modified for asymmetric wall shapes following Richey [18], hold in the present case. The composite solutions for u and v, formed from the inner and channel-flow solutions, and the equation for u_s are as follows:

$$u = 1 \pm \sqrt{\frac{2}{\gamma+1}(c_1 - f)} + \epsilon^2 \left[-f'' \frac{y^2}{2} + g_2' y + h_{2x} + \hat{\zeta}_{2\hat{x}} \right] + \cdots \qquad (77a)$$

$$v = \epsilon^2 \left[-f' y + g_2 \right] + \epsilon^{5/2} \hat{\zeta}_{2y} + \cdots \qquad (77b)$$

$$\hat{\zeta} = 0 \qquad \text{for} \qquad x < x_{s0} + \cdots \qquad (77c)$$

$$\hat{\zeta} = \frac{32\sqrt{(\gamma+1)u_{1u}}}{\pi^3}\left\{g_{20}' \sum_{n=1,3,5,\cdots}^{\infty} \frac{(-1)^{\frac{n+1}{2}}}{n^3} \exp\left(-\frac{n\pi\hat{x}}{2}\Big/\sqrt{(\gamma+1)u_{1u}}\right) \sin\frac{n\pi y}{2}\right.$$

$$\left. - f_0'' \sum_{n=2,4,6,\cdots}^{\infty} \frac{(-1)^{n/2}}{n^3} \exp\left(-\frac{n\pi\hat{x}}{2}\Big/\sqrt{(\gamma+1)u_{1u}}\right) \cos\left(\frac{n\pi y}{2}\right)\right\}$$

$$\text{for } x > x_{s0} + \cdots \qquad (77d)$$

$$\hat{x} = (x - x_{s0})\,\epsilon^{-1/2} \qquad (77e)$$

$$\frac{4}{(\gamma+1)}\frac{u_s}{\epsilon^2} = \frac{4k}{(\gamma+1)}\frac{dx_{s0}}{dt} = (h_{2x})_d + (h_{2x})_u - \frac{f_0''}{3} - u_{1u}^2 \qquad (77f)$$

where u_{1u} is given by Eq. (37a), with the positive sign, evaluated at $x = x_{s0} + \cdots$ and is thus a function of time, and where the subscript 0 indicates that the function is evaluated at $x = x_{s0}$. The equation for u_s/ϵ^2 shows only the first term in the expansion since that is all that is desired here. The composite solutions for u and v are uniformly valid through $O(\epsilon^2)$ in u and $O(\epsilon^{5/2})$ in v throughout a channel in which a shock wave occurs, as long as the throat or leading-edge regions are avoided.

If Eqs. (40a) and (76a) are used to find $(h_{2x})_d$ and $(h_{2x})_u$ and the results are substituted into Eq. (77f), one finds, finally, the differential equation for the position of the shock wave to lowest order, x_{s0}:

$$\frac{4k}{(\gamma+1)}\frac{dx_{s0}}{dt} = -\frac{1}{(\gamma+1)u_{1u}}\left[\frac{2\gamma}{3}(\gamma+1)u_{1u}^3 - c_{2u} + c_{2d}\right] \qquad (78)$$

where c_{2u} and c_{2d} are the values of $c_2(t)$ used upstream and downstream of the wave, respectively. If the example discussed previously is considered, where the flow is subsonic entering the duct and accelerates to supersonic velocity upstream of the shock wave, with pressure oscillations impressed upon the flow at $x_1 > x_s$, then Eqs. (44) and (45) hold for c_{2d} and c_{2u}, respectively; then Eq. (78) becomes

$$\frac{4k}{(\gamma+1)} \frac{dx_{s0}}{dt} = -\frac{1}{(\gamma+1)\, u_{1u}} \left[\frac{2\gamma}{3} (\gamma+1) u_{1u}^3 + N(t)\right] \tag{79a}$$

$$N(t) = c_{2d}^{(s)} + G_d(t) + G(0,t) - G(x_1,t) \tag{79b}$$

Although analytical solutions may be found for special wall shapes and forcing functions, the calculation of x_{s0} generally will involve numerical integration of this equation.

Motion of the Shock Wave Upstream of the Throat

According to Eq. (78) or (79a) the shock-wave velocity $dx_{s0}/dt \to \infty$ if the upstream fluid velocity $u_{1u} \to 0$, as occurs if the shock wave approaches the throat as the time t approaches a value t_*. Expanding Eq. (79a) as $x_{s0} \to 0$ and $t \to t_*$, and then integrating,

$$\frac{4k}{\gamma+1} \frac{dx_{s0}}{dt} = -\frac{N(t_*)}{(\gamma+1)^{1/2} \left[-f''(0)\right]^{1/2} x_{s0}} + \cdots \tag{80a}$$

$$x_{s0}^2 = \frac{(\gamma+1)^{1/2} N(t_*)}{2k \left[-f''(0)\right]^{1/2}} (t_* - t) + \cdots \tag{80b}$$

where $N(t_*) = c_{2d}(t_*) + G(0,t_*)$, in accordance with the definition (79b).

If an initial shock-wave position has been specified, the complete solution for $x_{s0}(t)$ can be regarded as known, and therefore t_* is known. When the shock wave is near the throat, a different form of solution is required. It is shown below that the shock-wave velocity increases in order of magnitude as it passes through the throat region and becomes still larger as it moves further upstream in the duct. The end results obtained from the analyses of this and the preceding section are expressions for the shock-wave velocity in terms of position at all points of an accelerating channel flow which is subjected to prescribed small oscillations in wall shape or back pressure.

The composite solution (65) written for the velocity in the region downstream of the shock wave gives

$$u = 1 - \epsilon \left(\frac{2}{\gamma+1}\right)^{1/2} \{f(0) - f(x) + \epsilon c_{2d}(t) + \epsilon G(x,t)\}^{1/2} + O(\epsilon^2) \qquad (81)$$

where in general $c_{2d}(t) \neq c_{2u}(t)$ and so $c_{2d}(t) \neq -G(0,t)$; i.e., $N(t) \neq 0$. As the shock wave moves upstream, the perturbation in fluid velocity behind the shock wave found from this solution is $O(\epsilon^{3/2})$ when $x_s = O(\epsilon^{1/2})$:

$$u_d = 1 - \epsilon^{3/2} \left(\frac{2}{\gamma+1}\right)^{1/2} \{-\frac{x^2}{2} f''(0) + N(t)\}^{1/2} + O(\epsilon^2) \qquad (82)$$

The velocity just upstream of the shock wave, since $c_{2u}(t) + G(0,t) = 0$ and $f''(0) > 0$, is

$$u_u = 1 + \epsilon^{3/2} \left(\frac{2}{\gamma+1}\right)^{1/2} \left(\frac{x}{\epsilon^{1/2}}\right) \{-\frac{1}{2} f''(0)\}^{1/2} + O(\epsilon^2) \qquad (83)$$

The shock-wave jump conditions (30d, e), with $H = \frac{1}{2}(\gamma+1)/(\gamma-1)$, give

$$u_s = \frac{\gamma+1}{4}\{(u_u - 1) + (u_d - 1)\} + \cdots \tag{84}$$

The shock-wave speed $u_s \equiv dx_s/dT$ is therefore $O(\epsilon^{3/2})$, and the shock wave remains within a distance $x_s = O(\epsilon^{1/2})$ of the throat for a time interval $\Delta T = O(\epsilon^{-1})$. Since $t = k\epsilon^2 T$, the interval in t is small, namely $\Delta t = O(\epsilon)$. The shock wave therefore passes through the thin region $x_s = O(\epsilon^{1/2})$ at essentially a constant value of t. For matching with the downstream solution, as shown below, this constant has the value t_* obtained in Eq. (80b). Hence t is replaced by $t^* + O(\epsilon)$ in the function N(t), with u_s expected to be obtained as a function of $(t - t_*)/\epsilon$. Since $u_s = k\epsilon^2 dx_s/dt$, Eq. (84) becomes

$$k\epsilon^2 \frac{dx_s}{dt} = \epsilon^{3/2}\frac{(\gamma+1)^{1/2}}{4}\left\{\frac{x_s}{\epsilon^{1/2}}[-f''(0)]^{1/2} - [-\frac{x_s^2}{\epsilon} f''(0) + 2N(t_*)]^{1/2}\right\} + \cdots \tag{85}$$

The shock-wave position therefore has the form

$$x_s = \epsilon^{1/2}[-2N(t_*)/f''(0)]^{1/2}\,\xi^*_{s1}(t^*) + \cdots \tag{86a}$$

$$t^* = \frac{1}{4k}(\gamma+1)^{1/2}[-f''(0)]^{1/2}(t - t_*)/\epsilon \tag{86b}$$

Integration gives an implicit solution for $\xi^*_{s1}(t^*)$:

$$2t^* = 1 - (1 + \xi_{s1}^{*2})^{1/2} \{\xi_{s1}^* + (1 + \xi_{s1}^{*2})^{1/2}\}$$

$$- \ln \{\xi_{s1}^* + (1 + \xi_{s1}^{*2})^{1/2}\} \qquad (87)$$

As $\xi_{s1}^* \to \infty$ and $t^* \to -\infty$, this solution gives

$$\xi_{s1}^* = (-t^*)^{1/2} + \cdots \qquad (88)$$

The matching as $x_{s0} \to 0$ and $\xi_{s1}^* \to \infty$ requires that the largest terms of Eqs. (80b) and (88) be identical, when written in common notation. It is this condition, anticipated above, which requires that the time in Eq. (86b) be measured relative to the value t_* obtained from the downstream solution.

Next, as the shock wave moves upstream from the throat, for $\xi_{s1}^* \to -\infty$ and $t^* \to \infty$, it is found from the solution (87) that $\ln(-2\xi_{s1}^*) = 2(t^* - \frac{1}{4}) + \cdots$, or

$$x_s = -\left[\frac{N(t_*)}{-2\,e\,f''(0)}\right]^{1/2} e^{\frac{1}{2}(\gamma+1)^{1/2}[-f''(0)]^{1/2}\left\{\frac{t - t_*}{k\epsilon} - \frac{\ln(1/\epsilon)}{(\gamma+1)^{1/2}[-f''(0)]^{1/2}}\right\}} \qquad (89)$$

When $x_s = O(1)$, the channel solution (3~a) has the subsonic value (minus sign) both upstream and downstream of the shock wave, so that $u_{1d} = u_{1u}$, and the shock-wave velocity from Eq. (84) becomes $u_s = \frac{1}{2}(\gamma+1)\epsilon\, u_{1u} + \cdots$. That is, using also Eq. (21b), to order ϵ the shock wave travels upstream at a speed $u_s = u - a + \ldots$, and the order of magnitude of u_s has now increased further, so that $u_s = O(\epsilon)$. The solution for the position of the shock wave as it moves upstream from the throat, with $x_s = O(1)$, must have the form (89) as $x_s \to 0$. That is, the solution for $x_s = O(1)$ must match with the solution for $x_s = O(\epsilon^{1/2})$ at earlier times. This condition suggests a representation of the shock-wave position in terms of still another time coordinate, as follows:

$$x_s = x_{s0}^+ (t^+) + \cdots \tag{90a}$$

$$t^* = \frac{t - t_*}{k\epsilon} - \frac{\ln (1/\epsilon)}{(\gamma+1)^{1/2} [-f''(0)]^{1/2}} \tag{90b}$$

Substitution in the expression (84) for the shock-wave velocity gives

$$\frac{dx_{so}^+}{dt^+} = - \left(\frac{\gamma+1}{2}\right)^{1/2} \{f(0) - f(x_{s0}^+)\}^{1/2} \tag{91}$$

Since $f(0) - f(x_{s0}^+) \sim - \frac{1}{2} x_{s0}^{+2} f''(0)$ as $x_{s0}^+ \to 0$, the solution as $x_{s0}^+ \to 0$ and $t^+ \to -\infty$ has the form

$$x_{s0}^+ = A\, e^{\frac{1}{2} (\gamma+1)^{1/2} [-f''(0)]^{1/2} t^+} + \cdots \tag{92a}$$

$$A = - \left[\frac{N(t_*)}{-2\, e\, f''(0)}\right]^{1/2} \tag{92b}$$

where A is an integration constant. Comparison with Eq. (89) shows the exponents are identical, and the requirement that the solutions must match has been used to give the value of A. This result provides an initial condition for the differential equation (91).

The simplest special case arises for a duct of length $2x_1$, with $x_0 = -x_1$, having symmetric parabolic walls described by

$$f(0) - f(x) = f(0)\,\frac{x^2}{x_1^2} \tag{93}$$

where $f(\pm x_1) = 0$ and the choking condition (74) gives $f(0) = \frac{1}{2}(\gamma+1)u_{1\infty}^2$. Integration of the shock-wave velocity (91) shows that the solution (92), with $f''(0) = -(\gamma+1)\,u_{1\infty}^2/x_1^2$, remains correct for the entire shock-wave motion from the throat $x = 0$ up to the entrance $x = -x_1$. The time required for passage of the shock wave through this region is found to be

$$t - t_* = \frac{k\,x_1\,\epsilon}{(\gamma+1)\,(-u_{1\infty})}\,\ln\frac{2\,(\gamma+1)\,u_{1\infty}^2\,e}{\epsilon\,N\,(t_*)} \tag{94}$$

The shape (93) implies wedge-shaped leading and trailing edges, since $f'(\pm x_1) \neq 0$. A simple shape having cusped edges is given by

$$f(0) - f(x) = f(0)\,\sin^2\frac{\pi x}{2\,x_1} \tag{95}$$

where again $f(\pm x_1) = 0$ and $f(0) = \frac{1}{2}(\gamma+1)\,u_{1\infty}^2$. Eq. (91) can be integrated, and the integration constant is found by matching with Eq. (92) as $x_{s0}^+ \to 0$, now with $f''(0) = -\frac{1}{4}\pi^2\,(\gamma+1)\,u_{1\infty}^2/x_1^2$. The shock-wave position and the time required for the shock wave to move upstream through the front half of the duct are found as

$$\tan\frac{\pi\,x_{s0}^+}{4\,x_1} = -\left[\frac{N\,(t_*)}{8\,(\gamma+1)\,e\,u_{1\infty}^2}\right]^{1/2} e^{\frac{(\gamma+1)\,\pi\,(-u_{1\infty})\,t^+}{4\,x_1}} \tag{96a}$$

$$t - t_* = \frac{2\,x_1\,k\,\epsilon}{(\gamma+1)\,\pi\,(-u_{1\infty})}\,\ln\frac{8(\gamma+1)\,e\,u_{1\infty}^2}{\epsilon\,N\,(t_*)} \tag{96b}$$

The results (94) and (96b) for different shapes show identical dependence on the parameters; only the numerical factors are different.

While the dependence of the solutions on the parameters is shown explicitly in these special cases, applications for arbitrary shape and motion of the walls, and arbitrary oscillations in back pressure, will require numerical solution. In the preceding derivations the shock-wave velocity u_s is found from Eq. (79a) for $x_s = O(1)$ downstream of the throat, from Eq. (85) for $x_s = O(\epsilon^{1/2})$ near the throat, and from Eq. (91) for $x_s = O(1)$ upstream of the throat. For numerical calculations some procedure is needed to indicate what solution is to be used at each particular location. It is probably most convenient to replace these three representations by two composite expressions, one for $x_s \geq 0$ and one for $x_s \leq 0$, and both uniformly valid to order ϵ^2. This formulation can be accomplished for $x < 0$ by adding the downstream and throat representations, and then subtracting the common part found for $x_s \ll 1$ and $x_s \gg \epsilon^{1/2}$; for $x_s < 0$ the upstream and throat solutions are added, and the common part for $|x_s| \ll 1$ and $|x_s| \gg \epsilon^{1/2}$ is subtracted. The result for $x_s > 0$ is

$$u_s = -\epsilon^2 \frac{\gamma}{3}\left[f(0) - f(x_s)\right] + \frac{\epsilon^2}{4}\left(\frac{\gamma+1}{2}\right)^{1/2} N(t)\left\{\frac{2^{1/2}}{x_s\left[-f''(0)\right]^{1/2}} - \frac{1}{\left[f(0) - f(x_s)\right]^{1/2}}\right\} + \epsilon^{3/2}\,\frac{(\gamma+1)^{1/2}}{4}\left\{\frac{x_s}{\epsilon^{1/2}}\left[-f''(0)\right]^{1/2} - \left[-\frac{x_s^2}{\epsilon} f''(0) + 2N(t)\right]^{1/2}\right\} + \cdots \quad (97a)$$

and for $x_s < 0$

$$u_s = -\epsilon \left(\frac{\gamma+1}{2}\right)^{1/2} [f(0) - f(x_s)]^{1/2} - \epsilon^{3/2} \frac{(\gamma+1)^{1/2}}{4} [-\frac{x_s^2}{\epsilon} f''(0) + 2N(t)]^{1/2}$$

$$- \epsilon \frac{1}{4} (\gamma+1)^{1/2} x_s [-f''(0)]^{1/2} + \cdots \qquad (97b)$$

where $u_s = dx_s/dT = k\epsilon^2 dx_s/dt$.

Discussion

The solutions presented here hold for unsteadiness caused by any combination of impressed oscillations of the wall shapes or the back pressure. As a result, they allow easy comparison of the effects of one as opposed to the other. Although the results presented here for the channel flow have a superficial resemblance to those given previously [15] for a study of channel flow with only oscillations in back pressure, there are some significant differences, as will be seen. In addition, the earlier analyses have been extended here to provide detailed solutions for the leading-edge and throat regions and for the shock-wave motion upstream of the throat.

The velocity of the shock wave, $u_s = k\epsilon^2 dx_{s0}/dt + \cdots$, and its position x_{s0} are given by Eq. (79a) and its integral, respectively. It is seen from Eq. (79b) that the forcing function for the shock motion $N(t)$ could have the same form for either oscillations in back pressure or in wall shape; i.e., either $G(x,t) = G(0,t) = G(x_1,t) = 0$ with G_d

equal to some function of time, or $G_d = 0$ with $G(x,t)$ such that $G(0,t) - G(x_1,t)$ equals the same function of time. Hence, insofar as the motion of the shock wave is concerned, one can find an equivalence between wall and back-pressure oscillations. This does not apply to the velocity and pressure distributions, however, as shown by the equation for h_{2x} (Eq. (40a)), which appears in u_2 and P_2. No matter what the boundary conditions may be, $G(x,t)$ depends upon x as long as a pitching motion of the walls is involved, so that $c_2(t) + G(x,t)$ also depends upon x; on the other hand, if *only* oscillations in back pressure occur, $G = 0$ and $c_2(t)$ depends only upon $G_d(t)$ and is independent of x. This point will be illustrated later in an example in which the differences are calculated for the flow upstream and downstream of the shock wave.

It is of interest to note that neither the solutions for the motion of the shock wave nor those for the distributions of velocity and pressure depend upon x_c, the center of rotation of pitching motion. This is shown in Eq. (79b) for the shock motion; the same result can be demonstrated for the velocity and pressure distributions by forming $c_2 + G$ upstream and downstream of the shock wave, using Eqs. (44) and (45) for c_{2u} and c_{2d}. In all cases, the difference between G functions, each evaluated at a different point, is found; hence, the $x_c \beta(t)$ term cancels. The net result is, then, that because it is evidently area differences which are important in determining flow conditions, the center of rotation is not important.

All of the solutions presented so far have been discussed in terms of application to ducts in which the entire upper and lower walls are oscillating. However, they may be applied also to the case corresponding to the flow through a channel with stationary walls but with a separated boundary layer downstream of the shock. That is, we imagine now an equivalent wall shape which is stationary ($G_\ell = G_u = 0$) for $x \le x_c$ say, where $x_c > x_s$, but which is hinged at $x = x_c$ and oscillates for $x > x_c$. It is seen that the solutions for the velocity components and thus the pressure, temperature, and density present no problems in this application and will not be pursued further here. However, the equation for the velocity of the shock wave exhibits interesting behavior which is worthy of discussion.

The equation which holds for dx_{s0}/dt is Eq. (78), where now the flow upstream of the shock wave is steady, but $u_{1u}(x_{s0})$ varies with time because it is evaluated at $x_{s0}(t)$. If we consider the case where the walls are oscillating as discussed above and the back pressure is oscillating at some point $x = x_1$, now with $x_1 > x_c$, and where the flow is choked at the throat, then this equation reduces to Eqs. (79) with $G(0,t) = 0$. Thus,

$$\frac{4k}{(\gamma+1)} \frac{dx_{s0}}{dt} = - \frac{1}{(\gamma+1)u_{1u}} \left[\frac{2\gamma}{3}(\gamma+1)u_{1u}^3 + c_{2d}^{(s)} + G_d(t) - G(x_1,t)\right] \quad (98)$$

This shows that the effect of the wall flapping at some point downstream of the shock wave has the same effect on the position of the shock wave as an oscillation in back pressure. Moreover, it is seen that it is the area change at the "end" of the duct, x_1, which is important insofar as the position of the shock wave is concerned, since if $G(x_1, t) = 0$, there is no effect whatsoever; moreover, the shape of the oscillating wall between $x = x_c$ and $x = x_1$ evidently has no effect on the position of the shock wave in any event. Finally, in this case, the motion of the shock wave is dependent upon x_c, the center of rotation of this flapping wall.

Example Calculations

It has been shown that for the orders of the parameters associated with problem 1, the effects of the oscillating walls and back pressures are interchangeable insofar as velocity and position of the shock wave are concerned, but not in the pressure and velocity distributions. In these distributions, the differences between the two cases are seen in the second-order terms, the first-order terms being independent of time. As an illustration of this feature of the solutions, reference is made to the results given in Ref. [15]; in that analysis, only oscillations in back pressure were considered, the walls being stationary. For example, Figs. 4a,b,c and 5 (Figs. 6a,b,c and 7 in Ref. [15], reproduced here for easy reference) show isotachs and wall and center-line pressure distributions at various times and u_s and x_{s0} as functions of time for the following conditions, where the notation has been changed to agree with that in the present report, and where it should be noted that the calculations were made for a symmetric channel, so that $f_u = f_\ell = f$:

$$
\begin{aligned}
f(x) &= 3 - \frac{18}{13}\, x^2 && 0 \le x \le 1 \\
&= -\frac{27}{13}\,(x-2)^4 - \frac{48}{13}\,(x-2)^3 && 1 \le x \le 2 \qquad (99) \\
&= 0 && x > 2 \\
G_d(t) &= 3\,(\gamma+1)\,\sin(2t) && t \ge 0 \\
&= 0 && t < 0
\end{aligned}
$$

$$\epsilon = 0.1 \qquad \gamma = 1.4 \qquad c_{2d}^{(s)} = -\frac{2\gamma}{3}\,(\gamma+1)\,(u_{1u}^{(s)})^3$$

$$\tau = 100 \qquad c_{2u}^{(s)} = 0$$

Here $u_{1u}^{(s)} = u_{1u}(x_{s0}^{(s)})$ where $x_{s0}^{(s)}$ is the location of the shock wave in steady flow, for which $G_d = 0$. Since no oscillations in wall shape were considered, $G_\ell = G_u = G = 0$. Thus the values of $c_2(t)$ to be used in h_{2x} (and thus in u_2, p_2, etc.), upstream and downstream of the shock wave, and the equation for dx_{s0}/dt (Eq. (79a)) are, for this case,

$$c_{2u} + G(x,t) = c_2(t) = 0 \tag{100a}$$

$$c_{2d} + G(x,t) = c_{2d} = c_{2d}^{(s)} + G_d(t) = c_{2d}^{(s)} + 3\,(\gamma+1)\,\sin(2t) \tag{100b}$$

$$\frac{4k}{(\gamma+1)}\,\frac{dx_{s0}}{dt} = -\frac{1}{(\gamma+1)\,u_{1u}}\left[\frac{2\gamma}{3}\,(\gamma+1)\,u_{1u}^3 + c_{2d}^{(s)} + 3(\gamma+1)\sin(2t)\right] \tag{100c}$$

where $k = (\tau\,\epsilon^2)^{-1}$. It is important to note that in this case, as illustrated by Eq. (100a), the flow upstream of the shock wave is steady.

If we now consider flow in a channel with symmetric oscillations of the wall, so that $G_\ell = G_u = G$, and with no oscillations in back pressure, so that $G_d = 0$, we see that Eqs. (79) and (100c) are the same, i.e., u_s and thus x_{s0} are the same, if

$$G(0,t) - G(x_1,t) = 3(\gamma+1)\sin(2t) \tag{101}$$

Referring to the definition of G in Eq. (4c), one sees that there can be no plunging motion of the walls, but that there is symmetric rotation; i.e., $G = (x - x_c)\beta(t) = G_\ell = G_u$. From Eq. (101), then, one finds the equation for $\beta(t)$, with the result that

$$G = -\frac{(x - x_c)}{x_1} 3(\gamma+1)\sin(2t) \tag{102}$$

Hence, if the walls oscillate with this function G, if $G_d = 0$ so that the back pressure is constant, and if all other conditions are as in Eq. (99), the shock-wave velocity and position will be as shown in Fig. 5.

From Eqs. (44) and (45), and (102), it is seen that the values of $c_2(t) + G(x,t)$ in h_{2x} are, upstream and downstream of the shock wave respectively,

$$c_{2u} + G(x,t) = -\frac{x}{x_1} 3(\gamma+1)\sin(2t) \tag{103a}$$

$$c_{2d} + G(x,t) = c_{2d}^{(s)} + \frac{(x_1 - x)}{x_1} 3(\gamma+1)\sin(2t) \tag{103b}$$

Hence, the isotachs and pressure distributions will vary from those shown in Figs. 4a,b,c by the time-dependent terms in Eqs. (103), which occur in u_2 and p_2, the terms of order ϵ^2.

Numerical computations were carried out with Eqs. (103) for $c_2 + G$ in h_{2x} and with all remaining conditions as in Eq. (99). In Figs. 6a,b,c, isotachs and pressure distributions downstream of the shock wave are shown at the same times as in Figs. 4a,b,c, for comparison. In Ref. [15], the flow upstream of the shock wave was steady so no isotachs were shown in that region; here isotachs upstream of the shock wave are not shown only because there is nothing with which they can be compared. Upon comparison of Figs. 4a,b,c and 6a,b,c, one sees that even though the position of the shock wave is the same, the isotachs and pressure distributions are quite different. Two points are of particular importance. First, attempts were made to find isotachs with the same values as those shown in Figs. 4a,b,c so that the differences in shape could be noted, but this was not always possible because velocities of the desired value did not occur; hence, significant differences between the two cases do exist. Second, it may be noted in Figs. 6a,b,c that the location of the sonic line ($P = 1$) does not coincide with the throat in steady flow, $x = 0$, whereas it does in Figs. 4a,b,c. The reason for this is, of course, that the computations shown in Figs. 6a,b,c are for the case where the walls are oscillating so that the position of the minimum area, and thus the sonic line, changes with time; this does not occur when unsteadiness is caused by variations in back pressure. It should be noted that the distances to the sonic line found from computations of the pressure distribution and shown in Fig. 6a were checked using Eq. (72), with excellent agreement.

In previous work described in the references, the motion of the shock wave through the subsonic flow upstream of the throat was not considered; hence numerical computations for u_s and x_{s0} stopped at the throat. Since this analysis has been completed now, with results given in Eqs. (97), it is possible to extend the numerical work presented in Refs. [12] and [15] for two example flows. Again, the results are valid for oscillations either in back pressure, with G_d as in Eq. (99) or in wall shape as in Eq. (102). The duct considered is that for which the wall shape is symmetric about the throat, so that $f(-x) = f(x)$ with $f(x)$ given in Eq. (99). The two cases chosen are cases (a) and (c) of Fig. 15 in Ref. [15], and the results for $x_{s0} > 0$ are taken from this report. For each case, $\epsilon = 0.1$, $\tau = 150$ and if oscillating back pressures are considered, $G_d = (\gamma+1)\, 4.5 \sin(2t)$. For case (a) $x_{s0}^{(s)} = 0.75$ and for case (b) $x_{s0}^{(s)} = 1.5$. The composite solution for $x_{s0} < 0$, Eq. (97b), was integrated numerically using as initial points those given in Ref. [15] (Fig. 15) at $x_{s0} = 0$. The results are shown in Fig. 7. In case (a), the shock wave moves upstream of the throat very rapidly, moves out of the duct and disappears. At a later time, it forms again at the throat, moves downstream and then upstream, again gaining velocity rapidly as it moves in the subsonic flow field. This process repeats itself as the back pressure (or walls) oscillate. The shock wave can never reverse its direction in the subsonic flow because the velocity relative to the wave must be supersonic. In case (c) a similar situation occurs as the wave moves upstream and out of the duct. As shown in Ref. [15], however, conditions are such that after the wave forms again at the throat and moves downstream, it never again moves upstream as far as the throat; this behavior is not shown here.

Problem (2): $\alpha = O(\epsilon^3)$, $\tau = O(\epsilon^{-1})$

In this problem, the amplitude of the oscillation of the walls remains of the same order as in problem (1), but the frequency increases such that τ becomes $O(\epsilon^{-1})$; oscillations in back pressure again occur in terms $O(\epsilon^2)$. For this case, then, we set

$$\alpha = \epsilon^3 \tag{104a}$$

$$\tau = (k\epsilon)^{-1} \tag{104b}$$

The equations for the wall shapes are those given in Eqs. (4).

Channel-Flow Solutions; Velocity and Position of the Shock Wave

Because the wall shapes and thus the boundary conditions are unchanged from problem (1), the result that the flow is steady in first order holds here also. Therefore, the shock-wave velocity is again $O(\epsilon^2)$, and from Eq. (7b) it is seen that, if $\tau = O(\epsilon^{-1})$, x_{s0} is constant and the shock-wave motion now has an amplitude $O(\epsilon)$; thus, $u_s = k\epsilon^2 dx_{s1}/dt + \cdots$. Also, because it is only in second order that a dependence upon time is found, we note that, using Eq. (8),

$$\phi_T = \frac{\epsilon^2}{\tau}\,\phi_{2t} + \cdots = O(\epsilon^3)$$

so that upstream of the shock wave the Bernoulli Eq. (10) is again such that $F(T) = \frac{1}{2}(\gamma+1)/(\gamma-1) + O(\epsilon^2)$. Using Eq. (24), one can see that since $u_s(u_d - u_u) = O(\epsilon^3)$, $H_d = H_u = \frac{1}{2}(\gamma+1)/(\gamma-1) + O(\epsilon^3)$; since only terms to $O(\epsilon^2)$ are required, the stagnation enthalpy may be considered constant throughout the flow field.

As mentioned previously, Richey [18] considered the problem of flow in an asymmetric channel with oscillations in back pressure only, in the terms of order ϵ^2, with $\tau = O(\epsilon^{-1})$. Because $\alpha = O(\epsilon^3)$, the oscillating walls enter the boundary conditions first for ϕ_{3y}, and thus affect only h_{2x}. The remaining terms in the solutions in the channel-flow region and the region enclosing the shock wave have the same form as those found by Richey [18]. As noted previously, these solutions have the same form as those found for problem (1), even through τ has a *different order for each* case, because in both problems the flow is steady in first order and the walls are stationary to second order. The common solutions and therefore those that hold for this problem are the solutions for ϕ_{1x} and ϕ_2, contained in Eqs. (37) and (39). Because only these outer terms and boundary conditions up to terms of order ϵ^2 are involved in the analysis of the inner region behind the shock wave, the solutions in that region are also unchanged from those presented in problem (1). Hence, the composite solutions given in Eqs. (77) hold here as well; moreover, the equation for u_s also holds, but it represents a differential equation for x_{s1}; i.e., Eq. (77f) becomes

$$\frac{4}{(\gamma+1)}\frac{u_s}{\epsilon^2} = \frac{4k}{(\gamma+1)}\frac{dx_{s1}}{dt} = (h_{2x})_d + (h_{2x})_u - \frac{f_0''}{3} - u_{1u}^2 \tag{105}$$

It remains only to find h_{2x} to complete the solutions to $O(\epsilon^2)$.

If the expansion, Eq. (8), for ϕ_1 and Eqs. (104) for α and τ are substituted into Eq. (22), with $H = H_\infty = \frac{1}{2}(\gamma+1)/(\gamma-1)$, and into the boundary conditions, Eq. (31), one finds the following governing equation and boundary conditions for ϕ_3, where substitutions have been made for the various derivatives of ϕ_2:

$$\phi_{3yy} = 2kh_{2xt} + (\gamma+1)\left[\phi_{1x}\left(-\frac{f''y^2}{2} + g_1' y + h_{2x}\right) + \frac{(2\gamma-1)}{6}\phi_{1x}^3\right]_x \tag{106a}$$

$$\phi_{3y}(x,1,t) = -\phi_{1x} f_u' + G_{ux} \tag{106b}$$

$$\phi_{3y}(x,-1,t) = \phi_{1x} f_\ell' - G_{\ell x} \tag{106c}$$

Next, if Eq. (106a) is integrated, the boundary conditions are applied, and one of the resulting equations is subtracted from the other, one obtains the governing equation for h_2:

$$2kh_{2xt} + (\gamma+1)(\phi_{1x}h_{2x})_x = (\gamma+1)\left[\phi_{1x}\frac{f''}{6} - \frac{(2\gamma-3)}{6}\phi_{1x}^3 + \frac{G}{(\gamma+1)}\right]_x \tag{107}$$

where f and G are defined in Eqs. (37b) and (40b), and where $(\gamma+1)\phi_{1x}\phi_{1xx} = -f'$ has been used in deriving Eq. (107). Upon comparing this equation with Eq. (40a), the expression defining h_{2x} in problem (1), one sees that the effect of τ being smaller by $O(\epsilon)$ is to introduce the time derivative of h_{2x} into the governing equation.

It is possible to separate the effects of oscillations in back pressure and wall shape by making the following substitution

$$\phi_{1x} h_{2x} = \phi_{1x} \frac{f''}{6} - \left(\frac{2\gamma-3}{6}\right)\phi_{1x}^3 + M\left(t - \frac{2k}{\gamma+1}\int_{x_1}^{x} \frac{dx}{\phi_{1x}}\right) + R(x,t) \qquad (108)$$

where $R(x,t)$ contains the effects of the wall oscillations and the functional form of M depends on the back pressure at $x = x_1$, as well as upon $R(x_1,t)$. That is, if Eq. (108) is used for h_{2x} and the condition on pressure is applied at $x = x_1$, then, as in problem (1),

$$M(t) + R(x_1,t) = c_{2d}^{(s)} + G_d(t) \qquad (109)$$

If Eq. (108) is used for h_{2x} in Eq. (107), one finds the following governing equation for $R(x,t)$

$$\frac{2k}{(\gamma+1)} R_t + \phi_{1x} R_x = \phi_{1x} G_x = \phi_{1x} \beta \qquad (110a)$$

$$\beta = \frac{\beta_u + \beta_\ell}{2} = \beta(t) \qquad (110b)$$

where β_u and β_ℓ, defined in Eq. (4c), are associated with the pitching motion of the walls. Thus, it is seen that the equation for $R(x,t)$ along characteristics and the equation defining the one family of characteristics are, respectively,

$$\frac{dR}{dt} = \frac{(\gamma+1)}{2k} \phi_{1x} \beta \qquad (111a)$$

$$\frac{dx}{dt} = \frac{(\gamma+1)}{2k} \phi_{1x} \qquad (111b)$$

Since it does not appear possible to find analytical solutions for general wall shapes, numerical computations must be employed. However, for special wall shapes, it is possible to find analytical solutions which illustrate the important features of the flow field.

Analytical Solutions for a Simple Case

Analytical solutions for special cases, while lacking generality, are valuable in giving a qualitatively correct indication of the physical effects to be expected in general, and in showing the manner in which these effects will change as the important parameters are varied. Such solutions can provide an important guide for numerical calculations, serving as reference solutions exhibiting the trends which should be anticipated. In the case considered here, the flow accelerates through sonic speed at the throat, and a shock wave is present further downstream. The back pressure is held constant, but the walls undergo a simple harmonic symmetric pitching motion. The replacement $\beta(t) = \beta_0 e^{it}$ is made in Eq. (4c), which becomes

$$G_{u,\ell} = (x - x_c)\, \beta_0 e^{it} \tag{112}$$

where $t = \epsilon k T$, so that the (scaled) frequency implicit in Eq. (112) is k. This form of time dependence is particularly significant because an arbitrary periodic function of time can, of course, be represented by a Fourier series, and the problem formulated above is linear with respect to the wall motion, so that solutions for more general periodic functions can be constructed by superposition of solutions having the form derived here. To simplify the problem as far as possible, the walls are taken to be symmetric, with the parabolic shape $f(0) - f(x) = \frac{1}{2}(\gamma+1)(u_{1\infty}\,x/x_1)^2$ for $-x_1 < x < x_1$ as given by Eq. (93), where $f(0) = \frac{1}{2}(\gamma+1)\,u_{1\infty}^2$ by the choking condition (74a); here we have taken $x_0 = -x_1$. This shape might represent the first term in a series of polynomials representing a more general wall shape. Analytical solutions may, however, be impractical for more complicated shapes, especially because the shape function f(x) enters the solutions in a non-linear way.

For the case considered, the channel-flow solution for ϕ_{1x} is given by Eq. (37a), with $c_1 = f(0) = \frac{1}{2}(\gamma+1)\,u_{1\infty}^2$ from Eqs. (38) and (48) or (74a). The differential equation for h_{2x} is then found by substitution in Eq. (107). The results are, for $-x_1 < x < x_1$,

$$\phi_{1x} = \pm\,|u_{1\infty}|\,x/x_1 \tag{113a}$$

$$\begin{aligned} 2k\,h_{2xt} &\pm (\gamma+1)(|u_{1\infty}|/x_1)(x\,h_{2x})_x \\ &= \beta_0\,e^{it} \pm (\gamma+1)(|u_{1\infty}|/x_1)^3\left\{-\frac{1}{6}(\gamma+1) + \left(\frac{3}{2}-\gamma\right)x^2\right\} \end{aligned} \tag{113b}$$

where the upper signs are used everywhere upstream of the shock wave and the lower signs are used downstream. The solution ahead of the shock wave, required to be finite at the throat x = 0, is easily found by considering separately the steady-state part and the simple harmonic unsteady part. The result is

$$h_{2x} = \frac{1}{6}\left(\frac{u_{1\infty}}{x_1}\right)^2 \{-(\gamma+1) + (3-2\gamma)\,x^2\}$$

$$+ \frac{\beta_0\, x_1}{(\gamma+1)\,|u_{1\infty}|\,(1+\nu^2)^{1/2}}\; e^{i(t-\chi)} \tag{114a}$$

$$\nu = \tan\chi = \frac{2k\,x_1}{(\gamma+1)\,|u_{1\infty}|} \tag{114b}$$

If a is evaluated from Eq. (21b), it is seen that the speed $|u-a|$ at which small disturbances travel upstream becomes $u_\infty - a_\infty = \frac{1}{2}(\gamma+1)\epsilon\, u_{1\infty} + \cdots$ outside the channel. The parameter ν is a reduced frequency based on this velocity and on the frequency $k\epsilon$ and the length x_1. The phase lag angle χ increases from zero to $\pi/2$ as ν increases from zero to infinity, and χ increases as the magnitude of the wall curvature $\frac{1}{2}(\gamma+1)u_{1\infty}^2/x_1^2$ decreases.

As for problem (1), a local leading-edge solution is required in a thin vertical strip containing the edges and having width $O(\epsilon^{1/2})$. In the present example, however, the edges are wedge-shaped rather than cusped, and so a different solution is needed. The series representation for the perturbation potential now has the form

$$\phi = (u_\infty - 1)x + \epsilon^2\, \phi^*_{\frac{3}{2}}(x^*, y^*, t) + \cdots \tag{115a}$$

$$x^* = \frac{x + x_1}{(K\epsilon)^{1/2}}\quad,\qquad y^* = y \tag{115b}$$

The differential equation and boundary conditions for $\phi^*_{\frac{3}{2}}$ are

$$\phi^*_{\frac{3}{2}x^*x^*} + \phi^*_{\frac{3}{2}y^*y^*} = 0 \tag{116a}$$

$$\phi^*_{\frac{3}{2}y^*}(x^*, \pm 1, t) = \mp f'(-x_1) \tag{116b}$$

The perturbation velocity must decrease to zero as $x^{*2} + y^{*2} \to \infty$ outside the channel and must match with the channel solution as $x^* \to \infty$ with $-1 < y^* < 1$. The conformal transformation (55) again maps the flow onto the upper half of the complex ζ-plane. The solution is

$$(K\epsilon)^{1/2}(u - u_\infty) - i v = \frac{\epsilon^2}{\pi} f'(-x_1) \ln\left(1 - \frac{1}{\zeta^2}\right) + \frac{\epsilon^2 A}{2(\zeta^2 - 1)} + \cdots \tag{117}$$

where A is to be determined. As $z^* = x^* + i y^* \to \infty$ outside the channel, the right-hand side becomes $\epsilon^2 \{A - 2f'(-x_1)/\pi\}/2\pi z^* + \cdots$, the form for a source at $z^* = 0$, and at the edges $z^* = \pm i$ the solution has an inverse square-root singularity. Each of these effects, if nonzero, is stronger by a factor $O(\epsilon^{-1/2})$ than for the previous case of cusped edges. As $z^* \to \infty$ inside the channel,

$$(K\epsilon)^{1/2}(u - u_\infty) - i v = \epsilon^2 f'(-x_1)\left\{\frac{x + x_1}{(K\epsilon)^{1/2}} + \frac{1}{\pi} + i y\right\} - \frac{1}{2}\epsilon^2 A + \cdots \tag{118}$$

For the choked condition, the channel solution for $u = 1 + \phi_x$ will have no term of order $\epsilon^{3/2}$, as noted below. It follows that the matching with the channel solution requires that A have the constant value

$$A = \frac{2}{\pi} f'(-x_1) = -\frac{4}{\pi} \frac{u_{1\infty}^2}{x_1} \tag{119}$$

That is, $A < 0$ and the solution (117) evaluated as $z^* \to \pm i$ then shows that in this case there is an inward flow around the leading edges. The source strength $A - 2f'(-x_1)/\pi$ is zero; i.e., to the order considered here, no source term would appear in the solution for the outer flow. Thus the strength of the source seen by the outer flow has the same order of magnitude for both wedge-shaped and cusped leading edges. This is as expected, since the source strength should not depend on the edge details but on conditions further downstream. Again the channel solution has been completed without knowledge of the leading-edge solution, but near the channel entrance the flow is two-dimensional in the first approximation, and the leading-edge solution is essential for a correct flow description there. If desired, the more complicated term of order $\epsilon^{5/2}$ in ϕ could also be calculated, in the manner of Ref. [16].

In a thin region near the throat where $x = O(\epsilon)$, it can again be shown that a polynomial solution in x and y gives a correct flow description. As in the discussion preceding Eqs. (66) and (67), it is anticipated that a nonlinear differential equation may be needed, and the wall boundary condition must also be satisfied. These considerations suggest the expansions

$$u = 1 + \epsilon^2 u_1^* (x^*, y, t) + \cdots \tag{120a}$$

$$v = \epsilon^3 v_1^* (x^*, y, t) + \cdots \tag{120b}$$

$$x^* = \frac{x}{\epsilon} \tag{120c}$$

The differential equations and wall boundary conditions can be written as

$$v_{1y}^* = (\gamma + 1) u_1^* u_{1x^*}^* + 2 k u_{1t}^* \tag{121a}$$

$$v_{1x^*}^* = u_{1y}^* \tag{121b}$$

$$v_1^* (x^*, \pm 1, t) = \mp \{x^* f''(0) - G_x(0,t)\} \tag{121c}$$

where the wall shapes and wall motions have been assumed symmetric in Eq. (121c) but thus far are otherwise arbitrary. The solutions are

$$u_1^* = \left\{\frac{-f''(0)}{\gamma + 1}\right\}^{1/2} x^* + \frac{1}{2} f''(0) \left\{\frac{1}{3} - y^2\right\} + \lambda(t) \tag{122a}$$

$$v_1^* = -f''(0) x^* y + \frac{1}{6} (\gamma + 1)^{1/2} \{-f''(0)\}^{3/2} (y^3 - y) + y G_x(0,t) \tag{122b}$$

$$0 = 2k \frac{d\lambda}{dt} + (\gamma + 1)^{1/2} \{-f''(0)\}^{1/2} \lambda - G_x(0,t) \tag{122c}$$

In the present special case, $\lambda(t)$ is found as

$$\lambda = \frac{\beta_0 \quad e^{it}}{2ik + \dfrac{(\gamma + 1) \left| u_{1\infty} \right|}{x_1}} \tag{123}$$

Thus, as anticipated above, no term of order $\epsilon^{3/2}$ is needed in either the throat solution or the channel solution for u; if such terms had been introduced, the differential equations combined with the boundary and matching conditions would have shown that the terms must be identically zero.

The channel solution downstream of the shock wave requires additional terms from the solution to the homogeneous equation so that the downstream boundary condition can be satisfied. The result is

$$h_{2x} = \frac{1}{6}\left(\frac{u_{1\infty}}{x_1}\right)^2 \{-(\gamma+1) + (3-2\gamma)\,x^2\} - \frac{c_{2d}^{(s)}\,x_1}{(\gamma+1)\,|u_{1\infty}|\,x}$$

$$-\frac{\beta_0\,x_1}{(\gamma+1)\,|u_{1\infty}|\,(1+\nu^2)^{1/2}}\left\{1 - \frac{x_1}{x}\exp\left(i\nu \ln\frac{x}{x_1}\right)\right\} e^{i(t+\chi)} \qquad (124)$$

The constant $c_{2d}^{(s)}$ has the same meaning as in Eq. (43), with a value determined by the constant prescribed back pressure. If the back pressure were to vary with time as e^{it}, the integration constant chosen for the solution to the homogeneous equation would be different. It can be seen that this solution to the homogeneous equation corresponds to the function M in Eq. (108), having constant phase $t + \chi + \ln(x/x_1)$ along the characteristics $dx/dt = -x/\nu$ and hence describing disturbances which travel upstream at this speed from the channel exit. Again the particular solution gives the direct response to the motion of the walls. The in-phase part has a different sign than in the upstream solution because positive $\beta(t)$ now implies a velocity decrease, instead of an increase as was true upstream of the shock wave; this is more clear in the limit as $\nu \to 0$. The out-of-phase part retains the same sign as in the upstream solution, since it occurs as a consequence of the ϕ_{xt} term which has a damping effect on the fluid motion.

The motion of the shock wave now can be determined from Eq. (105). The functions $(h_{2x})_u$ and $(h_{2x})_d$ are obtained by evaluating the solutions (114a) and (124), respectively, at the first approximation $x = x_{s0}$ to the shock-wave position. The constant value x_{s0} is found by setting the time average of dx_{s1}/dt equal to zero, so that $c_{2d}^{(s)}$ has the same value as in the steady-flow case. The results are

$$c_{2d}^{(s)} = - \frac{2\gamma(\gamma+1)}{3} \left(\frac{|u_{1\infty}| x_{s0}}{x_1} \right)^3 \tag{125a}$$

$$k \frac{dx_{s1}}{dt} = \frac{\beta_0 x_1 e^{it}}{4 |u_{1\infty}| (1+\nu^2)^{1/2}} \left\{ -2i \sin \chi + \frac{x_1}{x_{s0}} \exp\left[i\left(\chi - \nu \ln \frac{x_1}{x_{s0}}\right)\right]\right\} \tag{125b}$$

The first term contains the effects of the upstream and downstream changes in flow velocity which result directly from the wall motion, and is 180° out of phase with the velocity $i\beta_0 (x - x_c) e^{it}$ of the wall motion. At a value of t when the channel width is decreasing upstream and increasing downstream, the flow velocity is increasing ahead of the shock wave and decreasing behind the shock wave, and the first term in dx_{s1}/dt implies that the shock wave is moving in the upstream direction. This term increases as the reduced frequency ν increases, since the lag in the shock-wave position increases as ν increases. The second term in dx_{s1}/dt represents the effect of flow disturbances which arise because the back pressure is prescribed and which travel upstream at a speed u - a from the channel exit. The phase lag $\nu \ln(x_1/x_{s0})$ is equal to the time Δt required for a disturbance to travel upstream from the exit $x = x_1$ to the shock wave $x = x_{s0} + \cdots$ at a speed $u - a = -\frac{1}{2}\epsilon(\gamma+1)|u_{1\infty}|\, x/x_1 + \cdots$.

The simple nature of the solutions for this special case permits a demonstration of the relationships between the solutions to problem (1) and those to problem (2), for which $\tau = O(\epsilon^{-2})$ and $\tau = O(\epsilon^{-1})$, respectively. The general principle is that the solutions to problem (1) evaluated as $\epsilon^2\tau \to 0$ should agree with the solutions to problem (2) evaluated as $\epsilon\tau \to \infty$. The present solutions can be investigated as $\epsilon\tau \to \infty$ by taking the limit as $k \to 0$. In this limit, the upstream solution (114a) for h_{2x} in fact has exactly the form (40a) with $c_2(t) + G(x,t) = \beta_0 x e^{it}$, which is zero at the throat $x = 0$ so that the condition (45) is satisfied. Similarly, as $k \to 0$ the downstream solution (124) for h_{2x} has the form (40a) with $c_2(t) + G(x,t) = c_{2d}^{(s)} + \beta_0 (x - x_1) e^{it}$, which is constant at the exit $x = x_1$ so that the condition (44) is satisfied. Thus the solutions for the flow velocities in problem (1) are limiting cases of the solutions for problem (2). This is not true, however, for the shock-wave velocity. As $k \to 0$, Eq. (125b) shows that the shock-wave velocity $u_s = dx_s/dT$ becomes

$$u_s = \epsilon^2 \frac{x_1^2 \beta_0 e^{it}}{4|u_{1\infty}|\, x_{s0}} \tag{126}$$

The shock-wave velocity in problem (1), as expressed by Eqs. (79), can be expanded as $\epsilon^2\tau \to 0$ by noting that u_{1u} approaches a constant value because the amplitude of the shock-wave oscillation becomes small. The value of $c_{2d}^{(s)}$ is again given by Eq. (125a), and substation for G shown that Eq. (79a) again gives the value (126) for u_s, as it must.

Problem (3): $\alpha = O(\epsilon^2)$, $\tau = O(\epsilon^{-1})$

In this problem, the amplitude of the oscillations of the wall is of the same order as the thickness of the wall (blade) itself, and is therefore an order of magnitude larger than in problems (1) and (2). The frequency is the same as in problem (2), i.e., $\tau = O(\epsilon^{-1})$. Thus, for this case, we set

$$\alpha = \epsilon^2 \tag{127a}$$

$$\tau = (k\epsilon)^{-1} \tag{127b}$$

The equations for the wall shapes are again those given in Eqs. (4).

Governing Equations, Channel-Flow Region

Just as in problems (1) and (2), application of the boundary conditions to the second-order terms leads to the derivation of the governing equation for the first-order term in the velocity potential ϕ_1. Because the boundary conditions do not involve first-order terms, ϕ_1 is again independent of y; however, because terms of order ϵ^2 in y_w are time-dependent, the first-order terms are no longer steady, but depend upon time. As a result, $u_s = O(\epsilon)$, and so from Eqs. (76) and (127b) $u_s = k\epsilon\, dx_{s0}/dt + \cdots$. That is, $x_{s0} = x_{s0}(t)$ and the magnitude of the oscillations of the shock-wave position is O(1); large excursions of the shock wave can occur. In addition, the oscillations in back pressure can occur in the term of order ϵ, rather than the term of order ϵ^2 as was the case in problems (1) and (2).

In order to derive the governing equations for the ϕ_i in Eq. (8), which again is used to represent the perturbation potential, it is necessary to find a^2; that is, it is necessary to evaluate F(T) in Eq. (10). For this case it is again convenient to use the energy equation written in terms of the stagnation enthalpy, Eq. (15), and to expand H as in Eq. (17a) with the governing equation for H_1 given by Eq. (17b). Comparing Eqs. (16) and 127b), we see that n = 1 in Eqs. (16) and (17a) for this case. In addition, Eqs. (13), (19), (20), and (21) hold here for P_1, P_2, ρ_1, H_1, F(T), and a^2, respectively. In Eqs. (20) and (21a), and thus in a^2 since $H - H_\infty = \epsilon^2 H_1$, the function g_1 occurs and must be evaluated both upstream and downstream of the shock wave. Although $g_1 = g_1(y,t)$ as a function of integration of Eq. (17b), it is seen from Eq. (21a) that it is at most a function of t. The evaluation of g_1 is more easily accomplished by considering the integration of Eq. (17b) at a given instant, i.e., by evaluating a definite integral. Thus, upstream of the shock wave,

$$H_1(x,t) - H_1(-\infty,t) = -k\,(\phi_{1t}(x,t) - \phi_{1t}(-\infty,t))$$

But $H_1(-\infty,t) = 0$ by definition, and we arbitrarily choose the unknown function of time in the potential to be zero, so $\phi_{1t}(-\infty,t) = 0$. Hence, upstream of the shock.

$$H_1 = - k \phi_{1t} \tag{128a}$$

$$g_1 = 0 \tag{128b}$$

In order to perform the same kind of calculation downstream of the shock wave, it is necessary to calculate the jump in H_1 across the wave. From Eqs. (24) and (30), it is seen that to the order desired,

$$H_d = H_u + u_s (u_d - u_u) + \cdots \tag{129}$$

where, again, the subscripts u and d refer to positions immediately upstream and downstream of the shock wave, respectively. If Eqs. (5a) and (17a) are used for the expansions of u and H, then Eq. (129) may be used to show that

$$H_{1d} = H_{1u} + \frac{u_s}{\epsilon} (u_{1d} - u_{1u}) \tag{130a}$$

$$H_{1u} = - k (\phi_{1t})_u \tag{130b}$$

where, since the flow is unsteady in first order, u_s must be $O(\epsilon)$, as mentioned previously. If now Eq. (17b) is integrated as a definite integral, at a given instant, from the shock wave to a point somewhere downstream, it is found that downstream of the shock wave

$$H_1 = H_{1d} - k (\phi_{1t} - (\phi_{1t})_d) \tag{131a}$$

$$g_1 = H_{1d} + k (\phi_{1t})_d \tag{131b}$$

where ϕ_{1t} will be calculated later.

If Eq. (21b), with $H - H_\infty = \epsilon^2 H_1$, and Eq. (8) are substituted into Eq. (9), with $T = \tau t$ and τ as in Eq. (127b), one obtains the following governing equations for ϕ_1, ϕ_2, and ϕ_3:

$$\phi_{1yy} = 0 \tag{132a}$$

$$\phi_{2yy} = 2k\phi_{1xt} + (\gamma + 1)\,\phi_{1x}\phi_{1xx} + 2\,\phi_{1y}\phi_{1xy} \tag{132b}$$

$$\phi_{3yy} = k^2\,\phi_{1tt} + 2\,k\,\phi_{2xt} - (\gamma - 1)\,\phi_{1xx}\,H_1 + 2k\,\phi_{1x}\phi_{1xt}$$

$$+ (\gamma + 1)(\phi_{1x}\phi_{2x})_x + (\gamma - 1)\phi_{1x}\phi_{2yy} + \frac{(\gamma+1)}{2}\,\phi_{1x}^2\phi_{1xx} \tag{132c}$$

The corresponding boundary conditions are found from Eq. (4), with $\alpha = \epsilon^2$, and Eqs. (5) and (31); thus

$$\phi_{2y}(x, 1, t) = -(f_u' - G_{ux}) \tag{133a}$$

$$\phi_{2y}(x, -1, t) = f_\ell' - G_{\ell x} \tag{133b}$$

$$\phi_{3y}(x, 1, t) = -\phi_{1x}(f_u' - G_{ux}) + k\,G_{ut} \tag{133c}$$

$$\phi_{3y}(x, -1, t) = \phi_{1x}(f_\ell' - G_{\ell x}) - k\,G_{\ell t} \tag{133d}$$

where $\phi_{ix} = u_i$ and $\phi_{iy} = v_i$.

Since $\phi_{1y} = 0$ at either boundary, then, from Eq. (132a), $\phi_{1y} = \phi_{1y}(x,t) = 0$ and so

$$\phi_1 = \phi_1(x,t) \tag{134}$$

With this result for ϕ_1, Eq. (132b) may be integrated to give

$$\phi_2 = \left((\gamma + 1)\,\phi_{1x}\phi_{1xx} + 2k\phi_{1xt}\right)\frac{y^2}{2} + g_2\,y + h_2(x,t) \tag{135}$$

After applying the boundary conditions on ϕ_{2y} given by Eqs. (133a) and (133c), one obtains two equations which may be combined to give

$$g_2(x,t) = \frac{1}{2}\left[f'_\ell - f'_u - G_{\ell x} + G_{ux}\right] \tag{136a}$$

$$2k\,\phi_{1xt} + (\gamma+1)\,\phi_{1x}\,\phi_{1xx} = -f' + G_x \tag{136b}$$

where f and G are defined in Eqs. (37b) and (40b), respectively. Eq. (136b) is seen to be the governing equation for ϕ_1; as mentioned earlier, this case is different from those considered in problems (1) and (2) because here ϕ_1 depends upon time. It may be noted that whereas the effects of the oscillating walls are felt in the governing differential equation itself through the function G, the oscillating back pressure is introduced either through initial or boundary conditions.

In order to complete the solutions to second order, it is necessary to find the governing equation for $h_2(x,t)$. As in the previous cases considered, this equation is found by integrating Eq. (132c) once, applying boundary conditions given by Eqs. (133b) and (133d), and subtracting one of the resulting equations from the other. The resulting equation is, after some rearranging,

$$\begin{aligned} 2k\,h_{2xt} + (\gamma+1)(\phi_{1x} h_{2x})_x = & -k^2\,\phi_{1tt} - \frac{k}{3}G_{xxt} + (\gamma-1)\,\phi_{1xx} H_1 \\ & + \frac{(\gamma+1)}{6}\left(\phi_{1x}(f-G)_{xx}\right)_x + (\gamma-1)\,\phi_{1x}\,(f-G)_x \\ & - 2k\,\phi_{1x}\,\phi_{1xt} - \frac{(\gamma+1)}{2}\,\phi_{1x}^2\,\phi_{1xx} - \phi_{1x}(f-G)_x + k\,G_t \end{aligned} \tag{137}$$

Equation (137) may be simplified somewhat by making substitutions for ϕ_{1tt} and ϕ_{1xt}. Thus, if Eq. (136b) is integrated over x, again in terms of a definite integral with $\phi_{1t}(-\infty,t) = 0$ as mentioned above, and the resulting equation is differentiated with respect to t, one finds that

$$k\,\phi_{1tt} = \frac{1}{2}\,G_t - \frac{(\gamma+1)}{2}\,\phi_{1x}\,\phi_{1xt} \qquad (138)$$

where $\phi_{1x}(-\infty,t) = u_{1\infty}$ = constant here and where F = G = 0 upstream of the channel, where $x < x_0$. If Eq. (138) is substituted into Eq. (137), and if Eq. (136b) is used for ϕ_{1xt} in the resulting equation, the governing equation for h_{2x} is found to be, finally,

$$\begin{aligned} 2k\,h_{2xt} + (\gamma+1)(\phi_{1x}\,h_{2x})_x &= \frac{k}{2}\,G_t - \frac{k}{3}\,G_{xxt} \\ &+ (\gamma-1)\,\phi_{1xx}\,H_1 + \left(\frac{\gamma+1}{6}\right)[\phi_{1x}\,(f-G)_{xx}]_x \\ &- \frac{(\gamma^2-1)}{4}\,\phi_{1x}^2\,\phi_{1xx} + \frac{(3\gamma-5)}{4}\,\phi_{1x}\,(f-G)_x \end{aligned} \qquad (139)$$

The last equation needed for solution of this problem to the order desired is that for the velocity and position of the shock wave. As in the problems considered previously, this relation is found from the jump conditions which hold across the moving shock wave. The relevant equations are (30d) and (30e), where u and H on the right-hand side of Eq. (30e) may be evaluated either both upstream or both downstream of the shock wave, as a result of Eq. (24); here conditions upstream of the wave are used. With Eqs. (5a) for u, (7b) for u_s, and

(17a) with $n = 1$ for H, the following equations are found for dx_{s0}/dt and dx_{s1}/dt:

$$\frac{4k}{(\gamma+1)} \frac{dx_{s0}}{dt} = u_{1d} + u_{1u} \tag{140a}$$

$$\frac{4k}{(\gamma+1)} \frac{dx_{s1}}{dt} = u_{2d} + u_{2u}$$

$$- 2 \left(\frac{\gamma - 1}{\gamma + 1}\right) \left(H_{1u} - u_{1u} k \frac{dx_{s0}}{dt} + \frac{k^2}{2} \left(\frac{dx_{s0}}{dt}\right)^2 \right) \tag{140b}$$

Thus, it is seen that to first order the jump conditions across the shock wave may be satisfied as long as Eq. (140a) is satisfied. The only point at issue is the initial condition in that x_{s0} must be specified at a given time; it is not possible, for example, to use Eq. (140a) to obtain the steady-state value of x_{s0}, when $dx_{s0}/dt = 0$. On the other hand, it is clear that Eq. (140b) cannot be satisfied in that $x_{s1} = x_{s1}(t)$ whereas u_{2d} and u_{2u} both depend upon y as well as upon time. Thus, the jump conditions cannot be satisfied to second order by the channel-flow solutions. As in previous cases, it will be necessary to consider an inner region about the shock wave which is now moving at $u_s = O(\epsilon)$. In this region, solutions will have to satisfy the local jump conditions across the shock wave and match with the outer channel-flow solutions. The inner region has not been analyzed as yet for this case. Hence the problem formulation described here allows solution only to first order, using Eqs. (136b) and (140a).

Simplified Formulation for a Special Case

Simplifications leading to an ordinary differential equation for the shock-wave motion can be achieved in a special case analogous to that considered for Problem 2. Although for the present problem a complete analytical solution can not be obtained, the example appears to provide a useful test case for numerical calculations, to show the general effects of changes in the parameters with numerical integration needed only for the shock-wave motion and not for the entire flow field.

In this special case the flow accelerates through sonic speed, at a location which in general no longer coincides with the location of the instantaneous minimum cross-section area, and a shock wave is present further downstream. The walls have the symmetric parabolic shape given by Eq. (93) and are given a simple harmonic pitching oscillation as shown by Eq. (112), where $t = \epsilon k T$ as before, so that the (scaled) frequency is again k. The difference is that the amplitude of the wall oscillations is $O(\epsilon^2)$ rather than $O(\epsilon^3)$; it follows that the sonic line and the shock wave undergo displacements which are $O(1)$ rather than $O(\epsilon)$.

With the choices (93) and (112) for the functions $f(x)$ and $G(x,t)$, the differential equation (136b) can be rewritten, with $\phi_{1x} = u_1$ for simpler notation, as

$$2 k u_{1t} + (\gamma + 1) u_1 u_{1x} = (\gamma + 1)(u_{1\infty}/x_1)^2 x + \beta_0 e^{it} \tag{141}$$

For an arbitrary right-hand side, this equation can be integrated numerically along characteristics $dx/dt = \frac{1}{2}(\gamma + 1)\, u_1/k$. In the present case, in order to obtain an analytical result, a coordinate transformation from x,t to r,s is carried out, such that s is measured along characteristics and so r is constant along characteristics; i.e., the value of r identifies a characteristic. If we take $\partial t/\partial s = 1$, the differential equation (141) is replaced by the system

$$\frac{\partial t}{\partial s} = 1 \tag{142a}$$

$$\frac{\partial x}{\partial s} = \frac{\gamma + 1}{2k}\, u_1 \tag{142b}$$

$$\frac{\partial u_1}{\partial s} = \frac{\gamma + 1}{2k} \left(\frac{u_{1\infty}}{x_1}\right)^2 x + \frac{\beta_0}{2k}\, e^{it} \tag{143c}$$

It is convenient to take s = 0 at t = 0, so that Eq. (142a) gives simply t = s. The general solutions for x and u_1 can be written as

$$x = A(r)\, e^{(s-r)/\nu} + B(r)\, e^{-(s-r)/\nu} - \frac{\beta_0 x_1^2}{(\gamma + 1) u_{1\infty}^2 (1 + \nu^2)}\, e^{is} \tag{144a}$$

$$u_1 = \frac{|u_{1\infty}|}{x_1} \{A(r)\, e^{(s-r)/\nu} - B(r)\, e^{-(s-r)/\nu}\} - \frac{i\, \beta_0\, x_1\, \nu}{(\gamma + 1)\, |u_{1\infty}|\, (1 + \nu^2)}\, e^{is} \tag{144b}$$

where A(r) and B(r) are to be determined from initial conditions for x(r,s) and $u_1(r,s)$, and ν is defined by Eq. (114b). For the flow downstream of the shock wave, $dx/dt < 0$ along characteristics, so that disturbances propagate upstream from the exit cross-section $x = x_1$ toward the shock wave, and initial conditions should then be specified at $x = x_1$. We can choose $r = s$ at $x = x_1$, so that the initial conditions are

$$u_1(r,r) = u_0 , \qquad x(r,r) = x_1 \tag{145a,b}$$

where u_0 is determined by the downstream pressure; we assume that the $O(\epsilon)$ term in the pressure at $x = x_1$ can be specified arbitrarily. The solutions for $x(r,s)$ and $u_1(r,s)$ downstream of the shock wave are then

$$\begin{aligned}\frac{x}{x_1} &= \frac{1}{2}\left\{1 + \frac{u_0}{|u_{1\infty}|} + \frac{x_1 \beta_0 (1+i\nu)}{(\gamma+1)\, u_{1\infty}^2 (1+\nu^2)}\, e^{ir}\right\} e^{(s-r)/\nu} \\ &+ \frac{1}{2}\left\{1 - \frac{u_0}{|u_{1\infty}|} + \frac{x_1 \beta_0 (1-i\nu)}{(\gamma+1)\, u_{1\infty}^2 (1+\nu^2)}\, e^{ir}\right\} e^{-(s-r)/\nu} \\ &- \frac{x_1 \beta_0}{(\gamma+1)\, u_{1\infty}^2 (1+\nu^2)}\, e^{is}\end{aligned} \tag{146a}$$

$$\begin{aligned}\frac{u_1}{u_{1\infty}} &= \frac{1}{2}\left\{1 + \frac{u_0}{|u_{1\infty}|} + \frac{x_1 \beta_0 (1+i\nu)}{(\gamma+1)\, u_{1\infty}^2 (1+\nu^2)}\, e^{ir}\right\} e^{(s-r)/\nu} \\ &- \frac{1}{2}\left\{1 - \frac{u_0}{|u_{1\infty}|} + \frac{x_1 \beta_0 (1-i\nu)}{(\gamma+1)\, u_{1\infty}^2 (1+\nu^2)}\, e^{ir}\right\} e^{-(s-r)/\nu} \\ &- i\nu\, \frac{x_1 \beta_0}{(\gamma+1)\, u_{1\infty}^2 (1+\nu^2)}\, e^{is}\end{aligned} \tag{146b}$$

For the flow ahead of the shock wave, the characteristics $dx/dt = \frac{1}{2}(\gamma+1)\,u_1$ have positive slope at points in the x,t plane where $u_1 > 0$ and negative slope at points where $u_1 < 0$, so that small disturbances are always propagated in a direction away from the sonic line where $u_1 = 0$. The first approximation to the location of the sonic line will be denoted by $x = x_*(t)$, where x_* is the value of x for which $u_1 = 0$. We will assume tentatively, subject to a later check for self-consistency, that $u_1 = O(x - x_*)$ as $x \to x_*$, just as for steady flow; that is, $\partial u_1/\partial x$ is neither zero nor infinite at $x = x_*$. It then follows that $dx/dt = O(x - x_*)$ along characteristics as $x \to x_*$ and therefore also $s \to -\infty$ as $x \to x_*$ along a characteristic. Thus necessarily $B(r) = 0$, and the solution for u_1 found from Eqs. (144a,b) is in fact simply proportional to $x - x_*(t)$:

$$\frac{u_1}{|u_{1\infty}|} = \frac{x - x_*(t)}{x_1} \tag{147a}$$

$$x_*(t) = -\frac{\beta_0\, x_1^2\,(1 - i\nu)}{(\gamma+1)\,u_{1\infty}^2\,(1+\nu^2)}\, e^{it} \tag{147b}$$

The result of course is consistent with the assumption made above. The instantaneous cross-section area of the channel has a minimum at the location $x = x_{min}$ where the right-hand side of Eq. (141) is zero. By comparison with Eq. (147b) it is seen that $x_* = x_{min}/(1 + i\nu)$, and so the sonic line no longer occurs at the cross-section having minimum area. Since $0 < \arg(1 + i\nu) < \pi/2$ for $0 < \nu < \infty$, the displacement of the sonic line lags the wall motion by a phase angle between zero and $\pi/2$; the amplitude in Eq. (147b) decreases as the frequency increases, with $x_* \to 0$ as $\nu \to \infty$.

A further check concerning the flow near the sonic line $x = x_*$ can be made by observing that the term $2k\phi_{xt}$ in the differential equation becomes of the same order as ϕ_{yy} when $x - x_* = O(\epsilon^{1/2})$. A local solution can be obtained in the form

$$\phi = \epsilon^2 \phi_2^* (x^*, y, t) + \cdots \qquad (148a)$$

$$x^* = \frac{x - x_*}{\epsilon^{1/2}} \qquad (149b)$$

One finds then

$$\phi^*_{2yy} + 2k \frac{dx_*}{dt} \phi^*_{2x^*x^*} = 0 \qquad (149a)$$

$$\phi^*_{2y} (x^*, \pm 1, t) = \pm \left((\gamma + 1) \frac{u_{1\infty}^2}{x_1^2} x_* + \beta_0 e^{it} \right) \qquad (149b)$$

The solution which matches correctly as $x^* \to \pm\infty$ with the channel solution evaluated as $x \to x_*(t)$ is

$$\phi_2^* (x^*, y, t) = \frac{1}{2} y^2 \left((\gamma + 1) \frac{u_{1\infty}^2}{x_1^2} x_* + \beta_0 e^{it} \right)$$

$$+ \frac{1}{2} \frac{|u_{1\infty}|}{x_1} x^{*2} + h_2 (x_*, t) \qquad (150)$$

where the function $h_2(x,t)$ is not yet known, but would be found by solution of Eq. (137). The result (150) simply says that the largest terms in the channel solution remain correct near the sonic line $x = x_*(t) + \cdots$, and that no special solution is needed there.

The shock-wave velocity dx_{s0}/dt is given by Eq. (140a) in terms of the flow velocities u_{1u} and u_{1d} immediately upstream and downstream. Since x is continuous at the shock wave, one more condition is available. To use this condition we can also define r to be continuous across the shock wave, so that a characteristic arriving at a given instant from the upstream side has the same value of r as the characteristic which reaches the shock wave from the downstream side at the same instant. From the continuity of $x(r,s)$, the function $A(r)$ in the upstream solutions for x and u_1 is then determined, and it is easily found that

$$\frac{u_{1u} - u_{1d}}{|u_{1\infty}|} = \left\{1 - \frac{u_0}{|u_{1\infty}|} + \frac{\beta_0 x_1 (1 - i\nu)}{(\gamma + 1) u_{1\infty}^2 (1 + \nu^2)} e^{iR}\right\} e^{(R-s)/\nu} \qquad (151)$$

The shock-wave velovity dx_{s0}/dt can be expressed in terms of the shock-wave position $R(s)$ if the solution for $x(r,s)$ is evaluated at $r = R(s)$ and then differentiated:

$$\frac{dx_{s0}}{dt} = \left.\frac{\partial x}{\partial s}\right|_{r=R} + \left.\frac{\partial x}{\partial r}\right|_{r=R} \frac{dR}{ds} \qquad (152)$$

With the substitution (142b) for $x(r,s)$, it is found from the first approximation (140a) to the shock-polar equation that

$$u_{1u} - u_{1d} = \frac{4k}{\gamma + 1} \left.\frac{\partial x}{\partial r}\right|_{r=R} \frac{dR}{ds} \qquad (153)$$

Substitution in Eq. (153) gives, finally, an ordinary differential equation for R(s)

$$\left\{1 - \frac{u_0}{|u_{1\infty}|} + \frac{\beta_0 x_1 (1-i\nu) e^{iR}}{(\gamma+1) u_{1\infty}^2 (1+\nu^2)}\right\}\left(\frac{ds}{dR} - 1\right)$$

$$= -\left(1 + \frac{u_0}{|u_{1\infty}|}\right) e^{2(s-R)/\nu} + \frac{\beta_0 x_1 e^{iR}}{(\gamma+1) u_{1\infty}^2}\left\{\frac{i\nu(1-i\nu)}{1+\nu^2} - e^{2(s-R)/\nu}\right\} \tag{154}$$

where it is understood that the real part is to be taken. For the simplest example the exit velocity is the same as the entrance velocity, so that $u_0 = u_{1\infty} < 0$; the remaining parameters are ν and $\beta_0 x_1/[(\gamma+1)u_{1\infty}^2]$. If an initial condition is specified, numerical integration of Eq. (154) would give the shock-wave location in the form $r = R(s)$; substitution in the solution (146a) for $x(r,s)$ then would allow calculation of $x_{so}(t)$. Presumably some sort of criterion for the existence of periodic solutions could thereby also be established. As a preliminary step, qualitative imformation about the solutions should be sought by consideration of limiting cases and by study of the singular points of the differential equation.

III. SUMMARY

The extension of previous work on unsteady transonic flow in two-dimensional channels to include effects of oscillating walls has proven to be relatively straightforward, for the cases considered, insofar as formulation of the problem is concerned. Thus, it has been possible to consider problems where the unsteadiness may be caused by any mixture of oscillations in wall position and back pressure. The essential difference between the two lies in the fact that when the walls are stationary, oscillations in back pressure cannot cause unsteadiness upstream of the shock wave, whereas when oscillations in wall position occur, the entire flow is time-dependent. Although it may be possible to find equivalent impressed oscillations for the walls and back pressure, such that the same shock-wave velocity and position result (e.g., problem 1), the distributions in flow properties are then not the same. Hence, as a general result, it is not possible to find equivalent impressed oscillations which give the same instantaneous flow fields.

The problems considered involve both small and large-amplitude shock-wave oscillations; in the latter case motion of the shock wave throughout the whole duct is allowed. In general, derivation of the governing equations for the flow properties has been carried out to the point that solutions valid to second order may be obtained. Where possible, analytical solutions have been presented. Where this was impossible, analytical solutions for special cases have been given and the problems formulated for numerical calculation.

Analysis of the thin region about the leading edge has extended previous analyses in that inlet problems, as well as straightforward channel flow, may now be considered. Although exterior cowl effects were not taken into account, because specific shapes would be needed, their addition would be straightforward.

One of the more interesting results found as a result of the detailed analysis of the throat region is the possibility of large excursions of the sonic line, and the fact that the sonic line need not, then, occur at the instantaneous position of minimum cross-sectional area.

The application to a duct with boundary layer which separates downstream of a shock wave such that the size of the bubble (and thus the effective channel wall shape) oscillates with time was considered in detail only for problem 1. However, this illustrated the method of solution well enough that further demonstrations were considered unnecessary.

Detailed numerical computations were given only for problem 1. Only example solutions with simple wall shapes and harmonic oscillations were shown for problem 2, because more complicated shapes and conditions would have necessitated extensive numerical work which would have added little understanding. Such is not the case for problem 3, where the simple solution presented allowed only calculation of the velocity field and not the velocity and location of the shock wave;

Moreover, this problem is important enough, involving unsteadiness in lowest order perturbation as it does, that it is recommended that detailed and fairly extensive numerical calculations should be carried out in future work.

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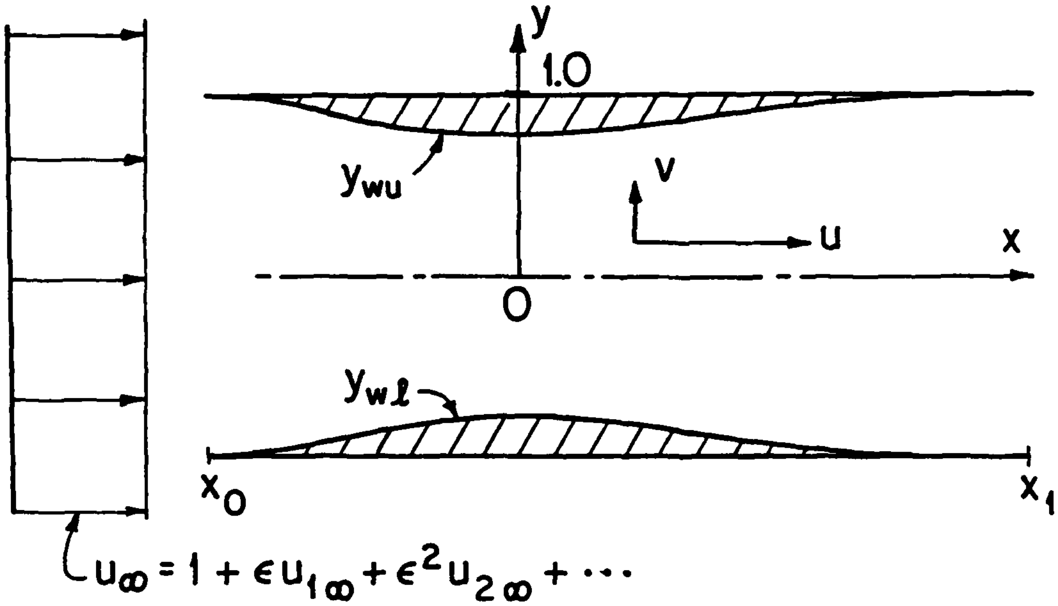

Figure 1. Coordinate system and notation used.

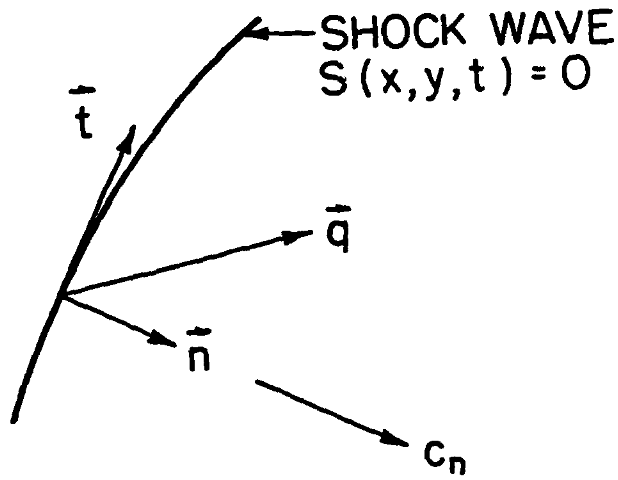

Figure 2. Coordinate system and notation for moving shock wave.

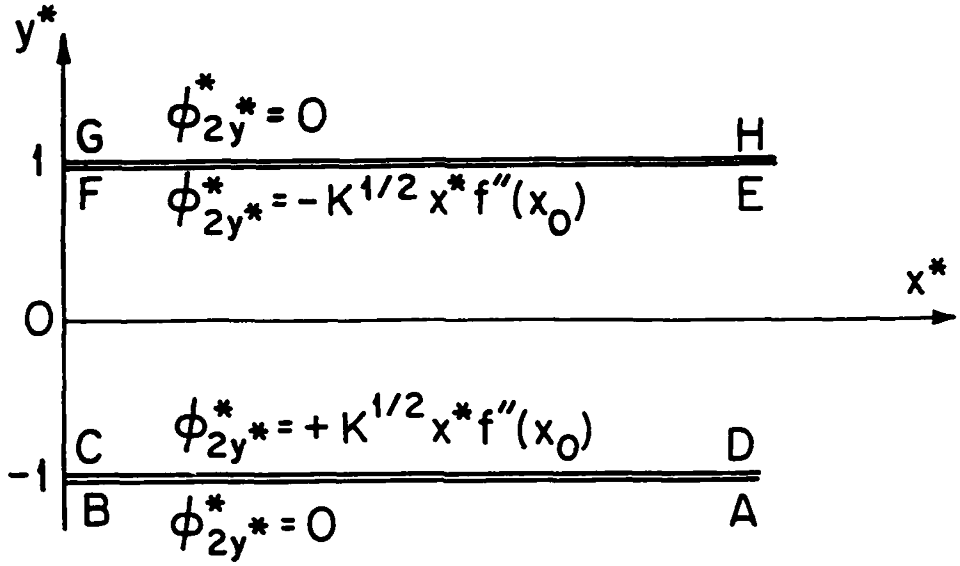

Figure 3a. Leading-edge region (z^* plane).

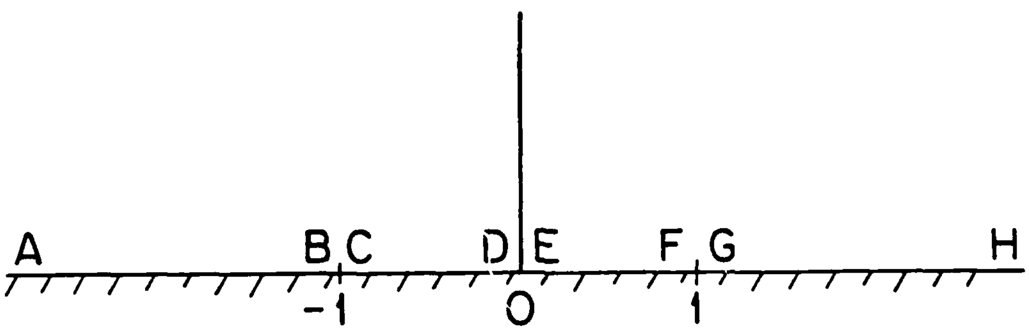

Figure 3b. Transformed leading-edge region (ζ plane)

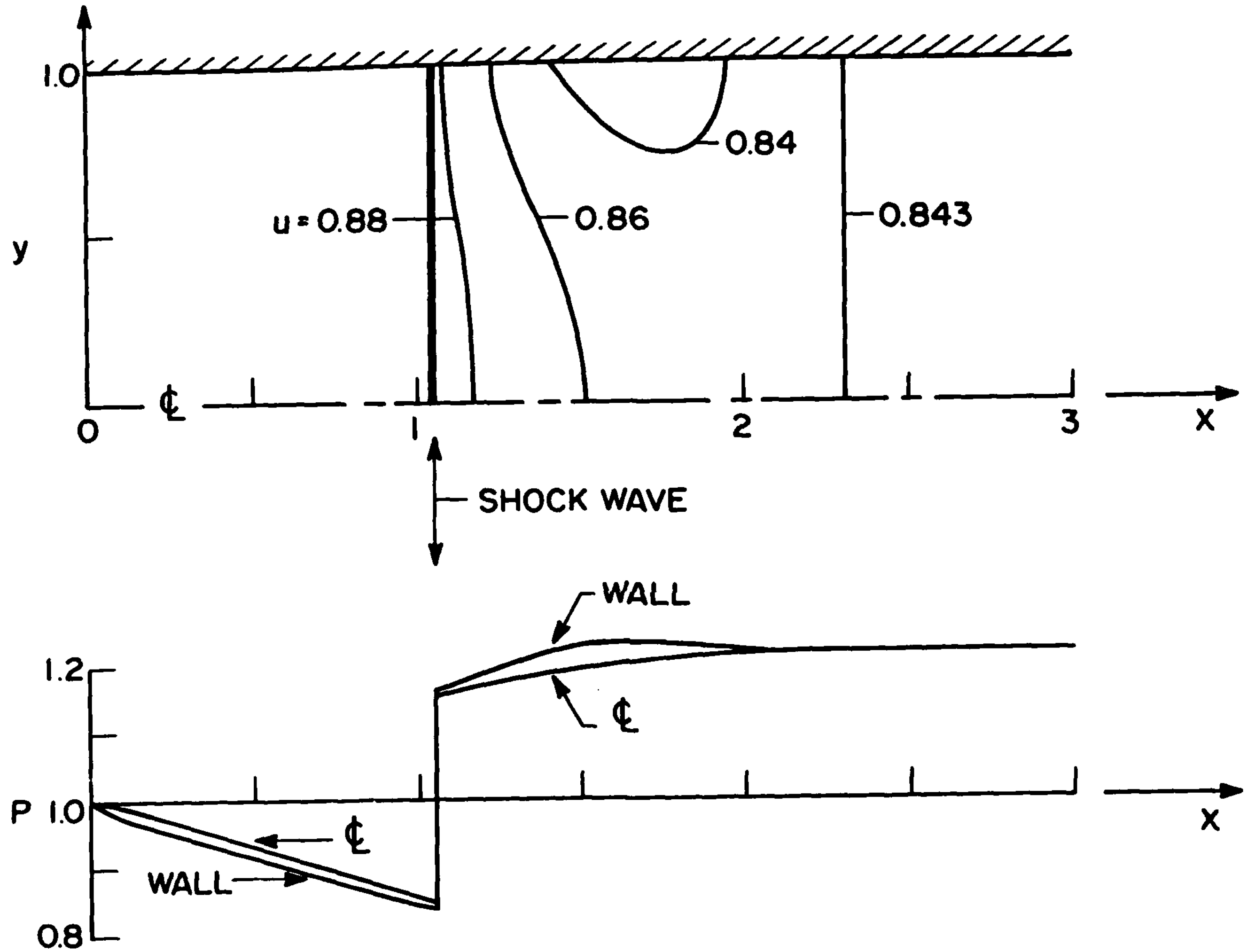

Figure 4. Isotachs and wall and centerline pressure distributions for conditions as given in Eq. (99); oscillations in back pressure only (Figures 6a,b, 7 in Ref. 7).

(a) t = 0.7854

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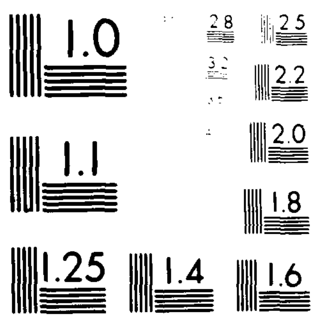
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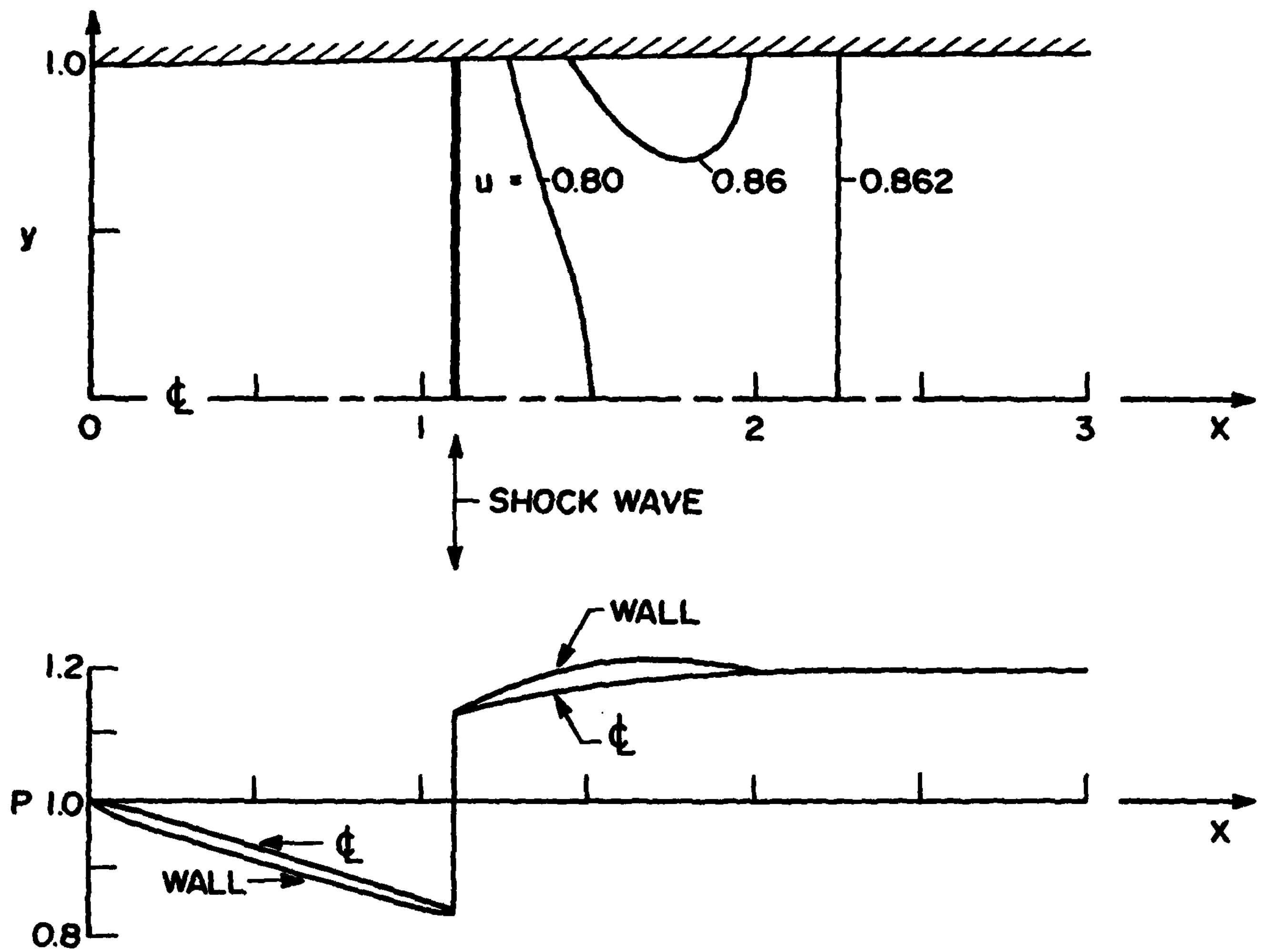

Figure 4. (b) t = 1.57079

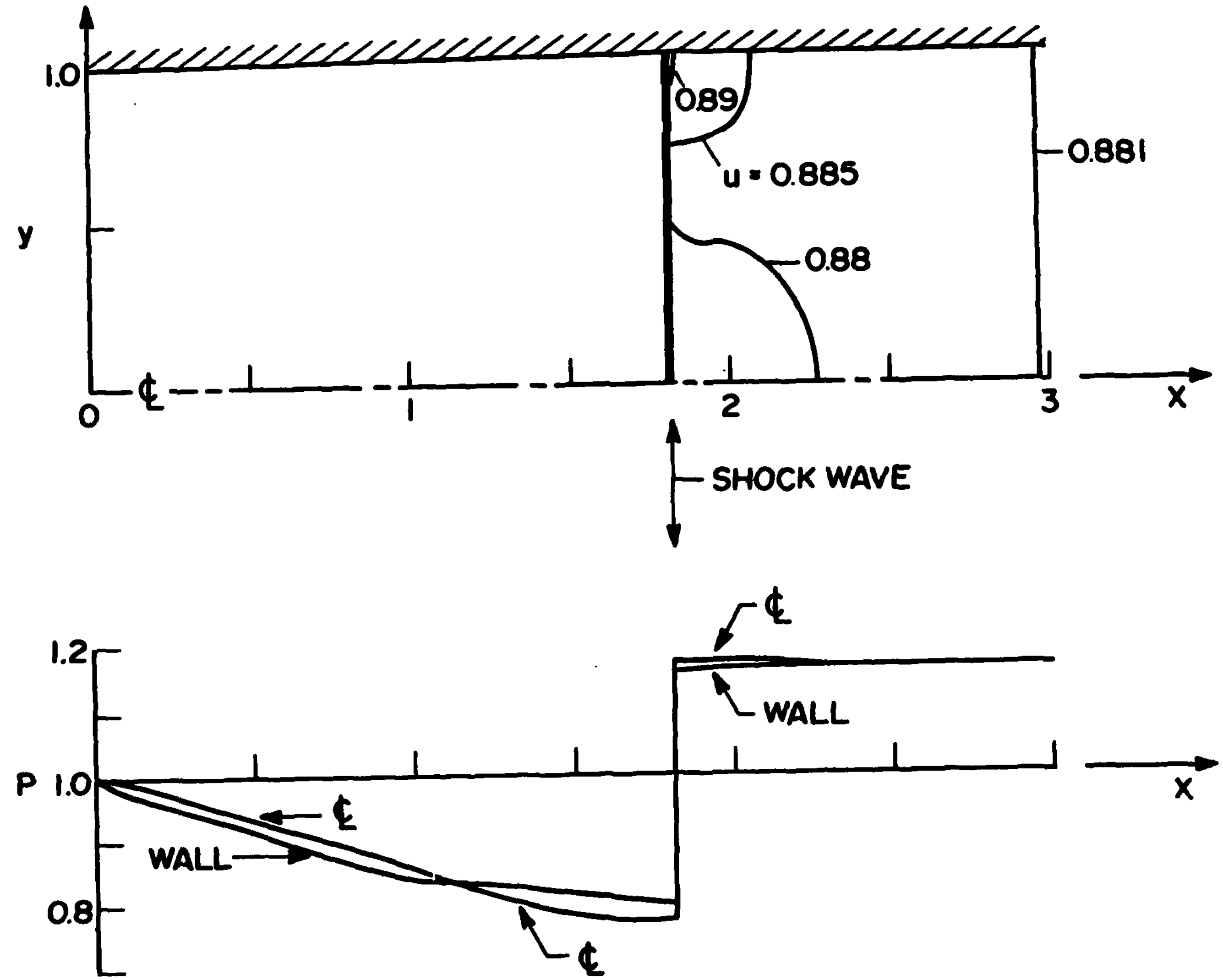

Figure 4. (c) t = 2.35618

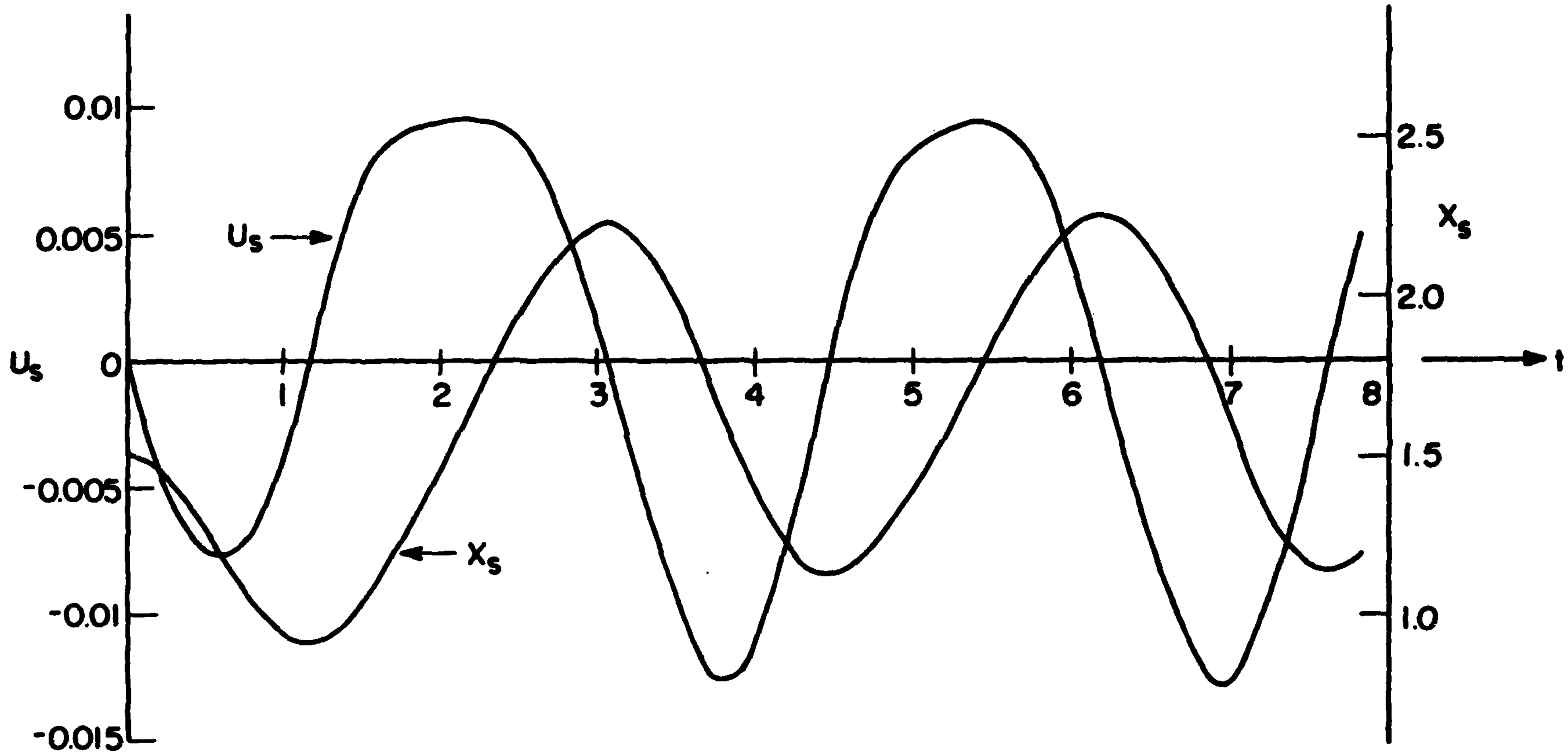

Figure 5. Shock wave velocity, $u_s = k\epsilon^2 dx_{s0}/dt + \cdots$ and position, $x_s = x_{s0} + \cdots$ as functions of time for conditions as given in Eq. (99) (Figure 7 in Ref. 15).

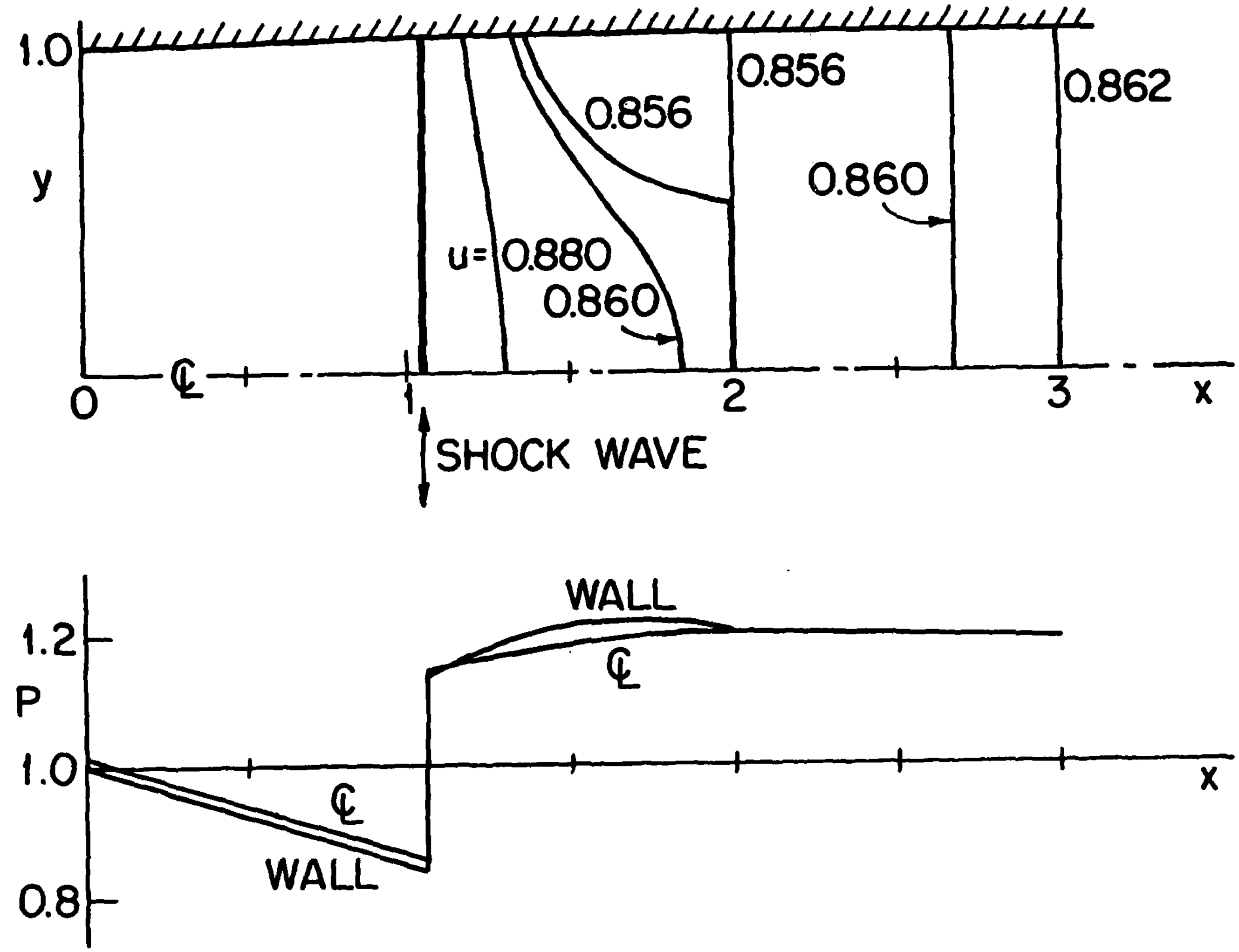

Figure 6. Isotachs and wall and centerline pressure distributions for $c_2 + G$ as in Eqs. (103) and all remaining conditions as in Eq. (99).

(a) $t = 0.7854$

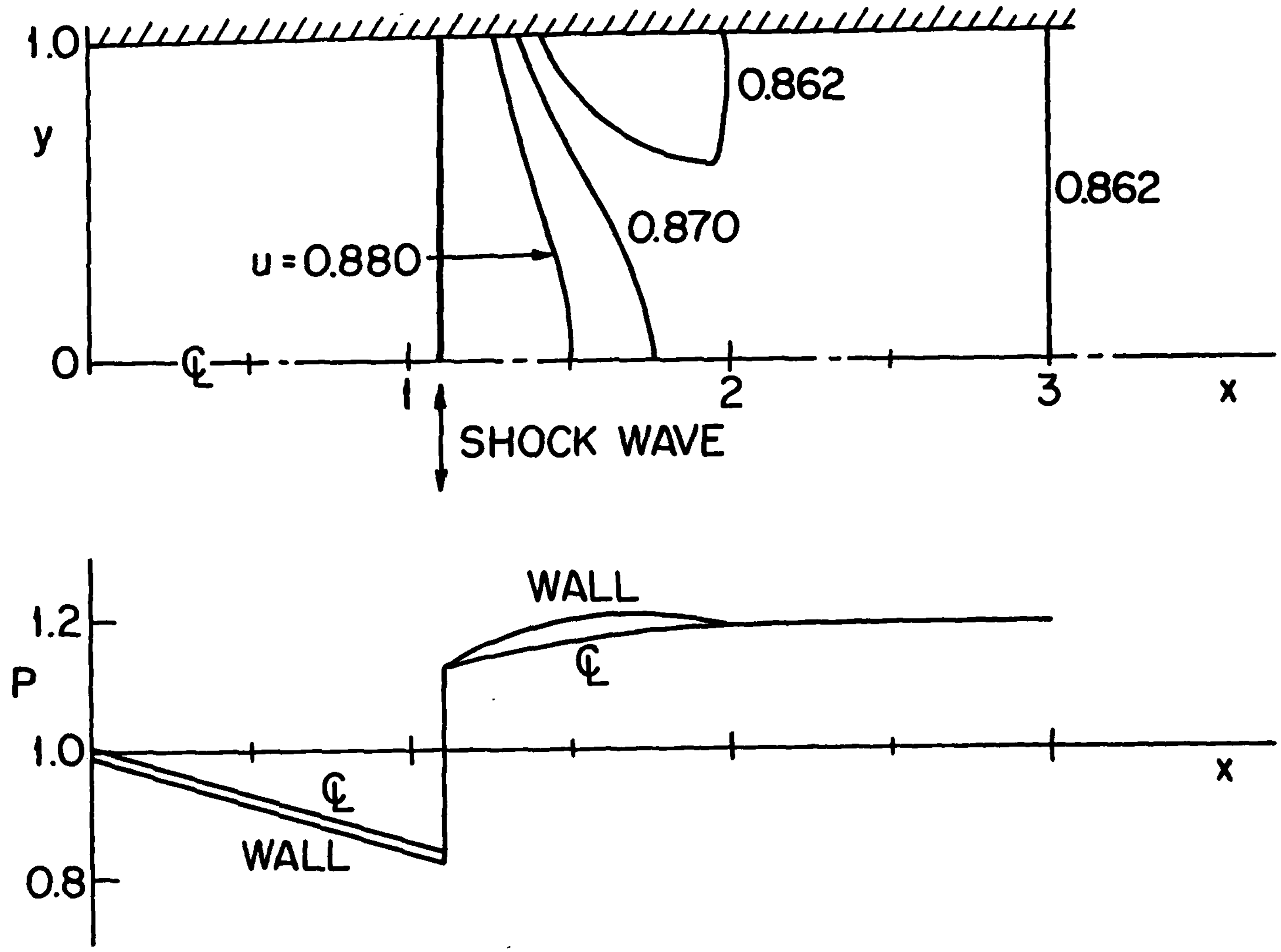

Figure 6. (b) t = 1.57079

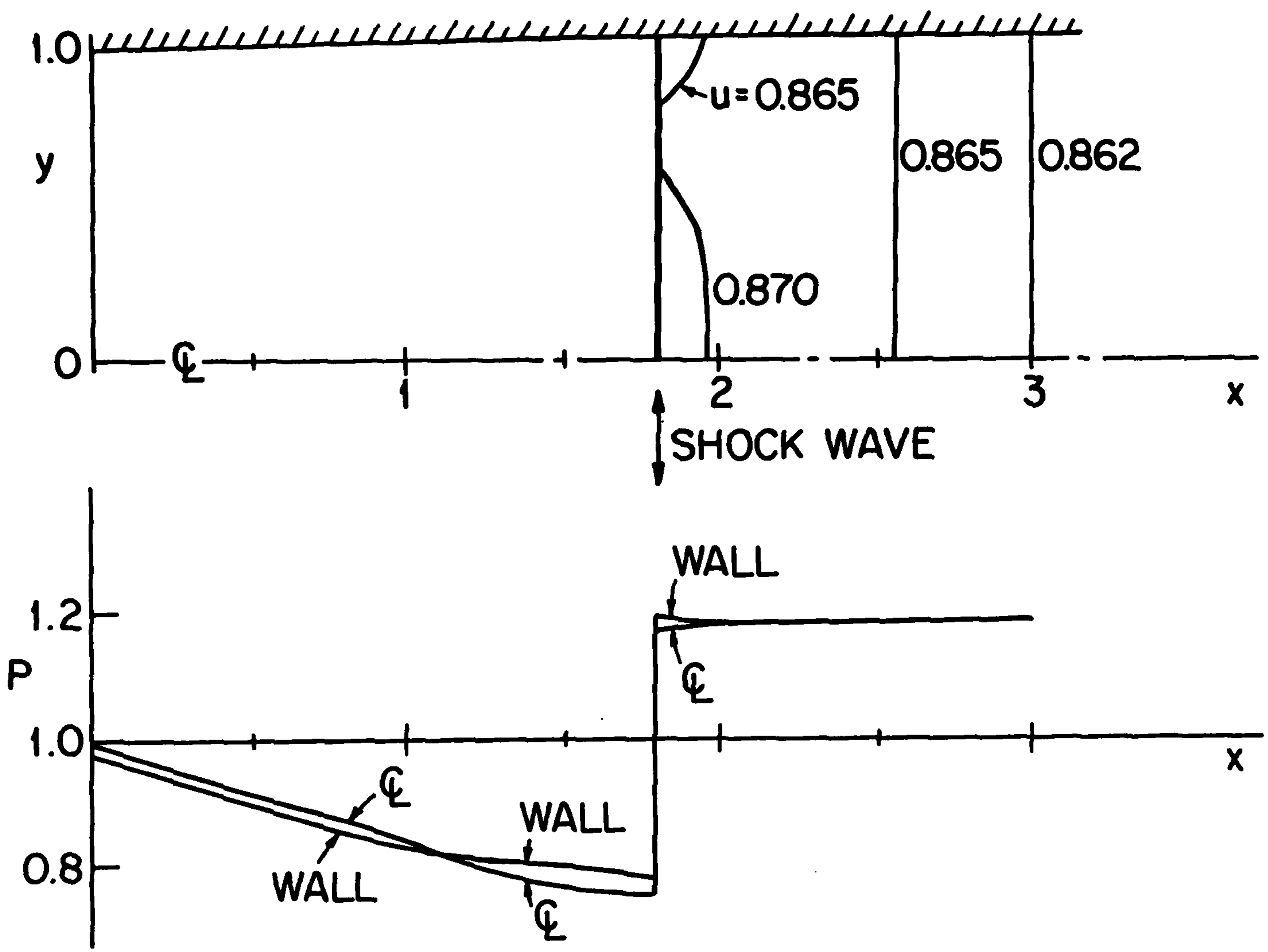

Figure 6. (c) t = 2.35618

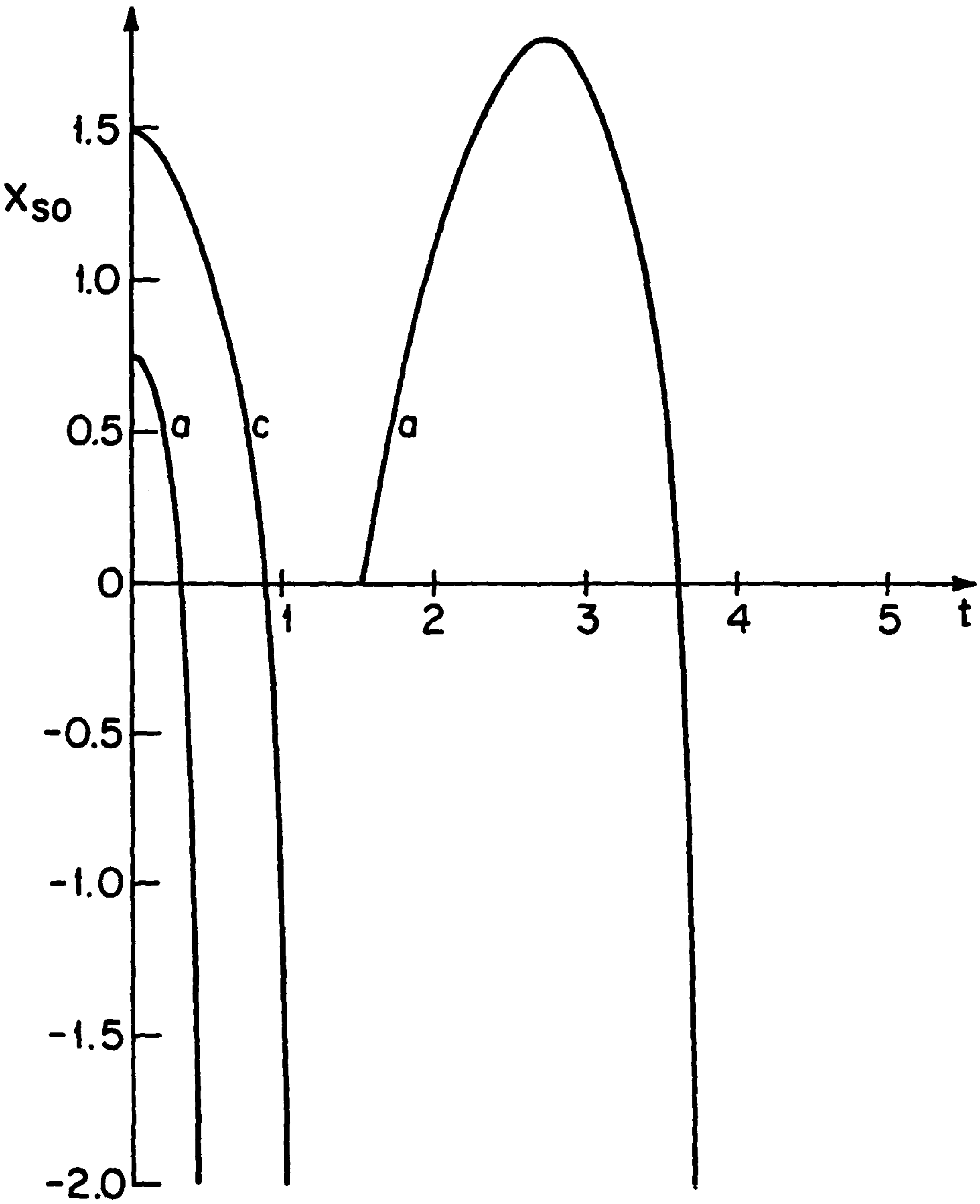

Figure 7. x_{s0} vs t from numerical integration of Eq. (98) for (a) $x_{s0}^{(s)} = 0.75$ and (b) $x_{s0}^{(s)} = 1.5$; in each case, $\epsilon = 0.1$, $\tau = 150$, $G_d = (\gamma + 1)(4.5) \sin (2t)$, $f(x)$ as in Eq. (99), and $\gamma = 1.4$. Solutions for $x_{s0} > 0$ from Figure 15 of Ref. 15.

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